Inductance Computations

Using

Partial Inductance Concepts

Albert Ruehli

November, 2007
Outline

● Partial Inductance Concepts
● Example Macromodels
● Speeding Up Using QR
Inductance Computation

Inductance Definition

- General loops with couplings
- $\Phi$ flux, $I$ current

\[ L_{11} = \frac{\Phi_1}{I_1} \quad L_{21} = \frac{\Phi_2}{I_1} \]
Inductance calculations

Partial Element Equivalent Circuits (PEEC)

- Loops are insufficient for general computations
- How do we model general geometries with loops?
- Many issues like skin-effect models are relevant

What Is a Partial Inductance?

- Calculate partial inductances of segments
- Building block approach for general geometries
- Circuit analysis to solve general case
Partial Inductance Concepts

Inductance of a Square Loop

- Break geometry into segments
- Start: Conventional formulation loop
- Use vector potential $A$

$$L_{loop} = \frac{\Phi}{I} = \frac{1}{I} \int_{a_{\Phi}} B \cdot da = \frac{1}{I} \int_{a_{\Phi}} (\nabla \times A) \cdot da$$  

(1)

$$B = \nabla \times A$$  

(2)
Partial Inductance Concepts

\[ L_{\text{loop}} = \frac{\Phi}{I} = \frac{1}{I} \int_{a_\Phi} (\nabla \times A) \cdot da \]  

(3)

Using Stoke’s Theorem

\[ \int_{a_\Phi} (\nabla \times A) \cdot da = \oint A \cdot d\ell \]  

(4)

\[ L_{\text{loop}} = \frac{1}{I} \frac{1}{a_c} \int_{a_c} \oint A \cdot d\ell da \]  

(5)
Partial Inductance Concepts (Cont.)

From last slide

\[ L_{loop} = \frac{1}{I} \frac{1}{a_c} \int_{a_c} \oint_{\ell} A \cdot d\ell da \]

The magnetic vector potential is given by

\[ A = \frac{\mu}{4\pi} \frac{I}{a'_c} \int_{a'_c} \oint_{\ell'} \frac{d\ell'}{|r - r'|} \cdot d\ell' da' \]
PEEC Inductance Derivation

\[ L_{\text{loop}} = \frac{\mu}{4\pi} \frac{1}{a'_c} \frac{1}{a_c} \int \int \int \int \frac{d\ell' \cdot d\ell}{|r - r'|} da'da \]

\[ L_{\text{loop}} = \sum_{j=1}^{4} \sum_{k=1}^{4} \frac{\mu}{4\pi} \frac{1}{a_k} \frac{1}{a_j} \int_{a_j} \int_{a_k} \int_{\ell_j} \int_{\ell_k} \frac{d\ell_k \cdot d\ell_j}{|r_j - r_k|} da_k da_j \]
Definition of Partial Inductance

\[ L_{p_{jk}} = \frac{\mu}{4\pi} \frac{1}{a_k} \frac{1}{a_j} \int_{a_j} \int_{\ell_j} \int_{a_k} \int_{\ell_k} \frac{d\ell_k \cdot d\ell_j}{|r_j - r_k|} da_k da_j \]

\[ L_t = \sum_{k=1}^{4} \sum_{j=1}^{4} L_{p_{jk}} \]  

(6)
Loop Inductance Calculations

Appy Kirchoff’s Laws for Circuits

- Voltage across partial inductances
- Currents through the partial inductances
Circuit Analysis using $L_p$s

General Circuit Analysis with Partial Inductances

- Can use conventional circuit analysis or Spice

$$
\begin{bmatrix}
L_{p11} & L_{p12} & L_{p13} & L_{p14} \\
L_{p21} & L_{p22} & L_{p23} & L_{p24} \\
L_{p31} & L_{p32} & L_{p33} & L_{p34} \\
L_{p41} & L_{p42} & L_{p43} & L_{p44}
\end{bmatrix}
\begin{bmatrix}
sI_1 \\
sI_2 \\
sI_3 \\
sI_4
\end{bmatrix}
=
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4
\end{bmatrix}
$$

$$
V = V_1 + V_2 + V_3 + V_4
$$  \hspace{1cm} (7)

$$
I = I_1 = I_2 = I_3 = I_4
$$  \hspace{1cm} (8)

$$
L = \frac{V}{sI} = \sum_{k=1}^{4} \sum_{m=1}^{4} L_{p_{km}}
$$  \hspace{1cm} (9)
Skin-Effect Model (VFI)

Skin-effect With \((L_p, R)\)PEEC Model
- Break up conductor cross-section into filaments
- Compute series resistance for each filament
- Compute partial inductance matrix

Conductor Proximity Effect
- Interaction of skin-effect in multiple conductors
- Strongest between close conductors
- Need to solve *global* interaction problem?

Model Simplification (Macromodels)
- Different approaches for simpler interaction models
- Simplified interaction models for distant conductors
- QR model shown below
Volume Filament Skin-Effect Model

One Simple Model for Skin-Effect

- Break up conductors into bars (filaments)
- Each bar (filament) has strong coupling to others
- Low frequency skin-effect: need to include resistances

Circuit Model for Each Conductor
Example L,R Skin-Effect

- Square cross section example
- Conductor spacing is same as size

SELF INDUCTANCE
(T=W=50 um, S=50 um, Length=2.5 cm)

RESISTANCE
(T=W=50 um, S=50 um, Length=2.5 cm)
Basic Observations

- Far spaced conductors
- Slow variation of $L_p$ with distance
- Very good approximate answer

\[ L_{pij}(d_{\text{max}}) < L_{pij}(\text{Exact}) < L_{pij}(d_{\text{min}}) \]

\[ L_{pij} \approx \frac{\int_{c_j} A \cdot dl_j}{d_{ij}} \approx \frac{\mu_o}{4\pi} \frac{l_i l_j}{r} \]
Proximity Macromodel

Simple Coupling Macromodel

- Each Coupling: SINGLE Mutual Partial Filament!
- Each Conductor: ONE Self Calculation
- Full Problem: \((N_p)^3\) Compute Time

November, 2007
Transmission Line Type Model

Example Model

- Usual model infinitely long line
- Need to use PEEC model for finite length!
- New insights by example
- Symmetrical Case
- Differential Mode Only

\[ I_2 = -I_1 \]
Transmission Line Type Model

Infinite Line has End Effect Compensation
- PEEC model of TL
- Simple coupling model
- Each section couples along infinite length
- Symmetry introduces reduced coupling

Finite Length Line, Corner
- Reduced inductance of finite length line
- Find error bound between the two models
- Can calculate L for examples like corner
TL Model for Infinite Length

- Section both conductors
- Compute partial inductances for sections
- Section size determines accuracy

\[
L_k = \frac{V_k - V_{k-1}}{sI} = 2 \sum_{m=-\infty}^{\infty} (Lp_{km} - Lp_{k'm})
\]

\[
L_{skm} = 2(Lp_{km} - Lp_{km'})
\]
Differential Coupling Decay

- Use section to section couplings
- How fast does coupling decay?
- Equation for differential couplings
- Distance between sections $\Delta x |k - m|

\[ L_{s_{km}} = 0.1 \Delta x q^2 / |k - m| \quad q := s / [\Delta x |k - m|] \]
### Differential Coupling Decay Example

<table>
<thead>
<tr>
<th>Section $N$</th>
<th>$Ls_{1N}$</th>
<th>$Ls_{1N}$ Approx.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.27765</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>0.013375</td>
<td>0.01</td>
</tr>
<tr>
<td>3</td>
<td>0.001309</td>
<td>0.00125</td>
</tr>
<tr>
<td>5</td>
<td>0.000159</td>
<td>0.0001563</td>
</tr>
<tr>
<td>7</td>
<td>4.66e-05</td>
<td>4.6296e-05</td>
</tr>
<tr>
<td>9</td>
<td>1.96e-05</td>
<td>1.953e-05</td>
</tr>
<tr>
<td>11</td>
<td>1e-05</td>
<td>1e-05</td>
</tr>
</tbody>
</table>
Section based Macromodels Without Couplings

Use Section Model

- Can we ignore coupling between sections?
- Check both types of models
- How large is error?
Macromodels Without Couplings

Transmission Line Section Model

- Total (coupled) TL
- Table: No coupling between sections results

<table>
<thead>
<tr>
<th>Sections $F$</th>
<th>$L_{1-F}$</th>
<th>$LS_{1-F}$ Macromodel</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.27765</td>
<td>0.27765</td>
</tr>
<tr>
<td>2</td>
<td>0.582049</td>
<td>0.555299</td>
</tr>
<tr>
<td>4</td>
<td>1.196842</td>
<td>1.110598</td>
</tr>
<tr>
<td>6</td>
<td>1.813189</td>
<td>1.665898</td>
</tr>
<tr>
<td>8</td>
<td>2.429942</td>
<td>2.221197</td>
</tr>
<tr>
<td>10</td>
<td>3.046943</td>
<td>2.776496</td>
</tr>
</tbody>
</table>
Macromodel Applications

- Many practical examples
- Semi Infinite Lines, corner etc.

![Diagram of Semi-Infinite Line Model](image-url)
Speed up for Partial Inductance Calculations

Can We Use Special Tricks?

- Simplifications examples above
- Stability: Partial inductance matrix is positive definite
- Cannot leave out coupling elements arbitrarily!
- Need other approaches

Mathematically Clean Approach

- Start with partial inductance matrix
- Matrix for subdivided problem is low rank
- Take advantage of low matrix rank
Large Lp Matrix in Block Form

General: Check Coupling factors for partitioning into blocks

- Find many couplings which are weak
- Coupling factors $\gamma \leq 0.25, 0.5$
- Max. inductive coupling: $L_c = \frac{L_{p12}}{\sqrt{L_{p11}L_{p22}}}$
- Far coupling factors are very small!
- Can also use distance-size related criteria
- Hence: easy to identify far coupled matrix blocks
Example Lp Matrix

- Weak coupling QR evaluation
- QR rank reduction for far couplings

Local coupling is dense

Far coupling, Low Rank Sections

November, 2007
Basics of QR Matrix Speed-Up (Gope, Jandhyala)

- Algorithm based on work by Kapur and Long
- Far partial element matrices are dense
- Coefficient values vary slow with distance
- Element matrix vector product is costly

QR Matrix Algorithm

- Far couplings rows/columns have less information
- This is indicated by rank $r$ of matrix
- QR exploits rank $r$ with sampled coupling equations
QR Matrix Compression Scheme

● Thinning of Coupling with QR Algorithm

\[ Q_k = \left[ A_k - \sum_{i=1}^{k-1} R_{ik} Q_i \right] / R_{kk} \]

\[ R_{ik} = Q_i^T A_k ; \quad k = 1 \cdots r \]
Conventional Partial Inductance Coupling
QR Compressed $L_p$ Coupling

\[
\begin{bmatrix}
  V_5 \\
  V_6 \\
  V_7 \\
\end{bmatrix}
= 
\begin{bmatrix}
  L_{p5a} & L_{p5b} \\
  L_{p6a} & L_{p6b} \\
  L_{p7a} & L_{p7b} \\
  L_p \\
\end{bmatrix}
\begin{bmatrix}
  \beta_{a1} & \beta_{a2} & \beta_{a3} & \beta_{a4} \\
  \beta_{b1} & \beta_{b2} & \beta_{b3} & \beta_{b4} \\
\end{bmatrix}
\begin{bmatrix}
  sI_1 \\
  sI_2 \\
  sI_3 \\
  sI_4 \\
\end{bmatrix}
\]
Summary and Conclusions

Inductance Computations

- General, flexible approach
- Use for analytical and numerical computations
- Partial element computations, $L_p$ matrix simplifications
- QR matrix vector product speed-up algorithm
- QR sampling speeds up matrix element computations
- Approximately $O(N \log N)$ solve time
- Inductance, quasi-static, full-wave PEEC applications

November, 2007