Kompendium

Cartesian Tensors

Marie Finnström
Håkan Gustavsson
Strömningslära
Inst TFM
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Introduction

The following pages are intended to cover the basic tensor material necessary for the course in Advanced Fluid Mechanics (MTM162) given in the 4th year of the undergraduate program at Luleå University of Technology. Tensor formalism is used in a variety of situations such as

- to formulate general physical laws
- to write the governing equations in a compact form
- to simplify complicated vector relations

The use of tensors is common to many branches of physics, mathematics and engineering. The material presented here should therefore be useful in other areas outside Fluid Mechanics.

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Marie Finnström
Håkan Gustavsson
1. Index notation

Coordinates are often denoted \( x,y,z \) but we shall here denote them \( x_1,x_2,x_3 \). Components of the stress tensor are then written as e.g. \( \sigma_{12} \) instead of \( \sigma_{xy} \). Here, the first index indicates the normal direction of a given surface and the second in which direction to project the stress. Each index can take the values 1,2,3 independently of the other. Thus, in each point there are 9 (=\( 3 \times 3 \)) stress components, which also can be arranged as a matrix

\[
\begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{bmatrix}
\]  

(1)

This may also be written symbolically as

\[ \sigma_{ij} \quad \text{where } i=1,2,3 \text{ and } j=1,2,3 \]

or even more compactly as

\[ \sigma_{ij} \quad \text{with } i,j=1,2,3 \]  

(2)

This way of writing stress components is denoted index notation.

Vectors, such as a force, are written in component form as

\[
\vec{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}
\]  

(3)

or with index

\[ F_i \quad \text{with } i=1,2,3 \]  

(4)

An index always runs through the values 1,2,3. We may therefore simplify the notation even further by omitting ‘with \( i,j=1,2,3 \)’ and simply write

\[ \sigma_{ij} \]

(5)

where it is understood that \( i,j=1,2,3 \).

1.1 The Einstein summation convention

The index notation is used to simplify the writing, and can be developed further for various types of expressions. One example is summation. The coupling between the
stress vector \((t_i)\) and the stress tensor \((\sigma_{ij})\) on a surface with normal direction \(\hat{n}\) is given by the Cauchy expression
\[
t_i = \sum_{j=1}^{3} \sigma_{ji} n_j.
\] (6)
Here, summation is over \(j\) which thus appears twice, both in \(\sigma\) and in \(n\). Thus we can introduce the convention (according to Einstein) that if the same index appears twice in the same term, summation is to be applied. This eliminates the need for writing summation signs. (6) can thus be written
\[
t_i = \sigma_{ji} n_j
\] (7)
This notation also applies to a single term so that \(\sigma_{ii}\) means
\[
\sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}
\] (8)

1.2 Free and dummy indices
An index which appears twice in the same factor, like in \(\sigma_{ii}\), means summation and is generally denoted a **dummy index**. This index may be substituted to any other symbol but never to a symbol that is already occupied. An index may never appear three or more times in one term. If an index appears single in an expression it is denoted a **free index**.

1.3 Tensors
The stress matrix and the force vector are examples of **tensors**. Index notation is therefore also denoted **tensor notation**. Except for the obvious simplicity of the tensor notation, tensors have also specific physical and mathematical properties. In brief, a tensor is a physical entity which is independent of any particular coordinate system but our description depends on the coordinate system we choose to express it in. As an example, the magnitude of a force and its direction do not depend on the coordinate system, but the components depend on our choice of description. A tensor is then defined as a quantity that follows certain transformation laws when expressed in different coordinate systems. In Appendix A we define the rules relevant for tensors.
Tensors are characterized by their number of free indices. This is denoted the **rank** of the tensor. Thus, a scalar has rank 0, a vector has rank 1 and the stress tensor has rank 2. The tensor $\sigma_{ji}n_j$ has one free index and is thus a vector. In an expression with many terms, the free index must be the same in all terms. Dummy indices can be denoted at will but must be chosen so not to be confused with other dummy indices. In particular, a dummy index must not be chosen the same as a free index!

### 1.4 Contraction

If two free indices are set equal in an expression, a **contraction** is performed. As an example, $b_{ij}^j$ is the contraction of $b_{ij}$. By a contraction, the rank of a tensor is lowered by two. Also, a summation is made over the contracted index.

### 1.5 Dot product

The dot, or scalar product can in the various notations (vector-, component- and tensor) be written as

\[ \vec{b} \cdot \vec{c} = b_jc_j = b_1c_1 + b_2c_2 + b_3c_3 \quad (9) \]

### 1.6 Multiplications

As a typical model for multiplications we consider the vector-matrix multiplication

\[ \vec{c} = M\vec{b} \quad (10) \]

where $\vec{c}$ and $\vec{b}$ are vectors and $M$ is a matrix. All quantities are also tensors. Writing out all elements in the matrix and the vectors, we have

\[
\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad (11)
\]

which after evaluations give

\[
\begin{align*}
  c_1 &= M_{11}b_1 + M_{12}b_2 + M_{13}b_3 \\
  c_2 &= M_{21}b_1 + M_{22}b_2 + M_{23}b_3 \\
  c_3 &= M_{31}b_1 + M_{32}b_2 + M_{33}b_3 
\end{align*} \quad (12)
\]

Using index notation this can compactly be written as
\[ c_i = M_{ij} b_j \]  \hspace{1cm} (13)

It is worth noting that the relative position of the terms in (13) is irrelevant so that we may write

\[ c_i = M_{ij} b_j = b_j M_{ij} \]  \hspace{1cm} (11)

In vector-matrix notation the order is important, however. Then, we use transpose to denote the vector \( a^T = (a_1, a_2, a_3) \) and thus the two expressions \( a^T M \) and \( Ma \) mean different things. In index notation they correspond to \( a_i M_{ji} \) and \( M_{ij} a_i \), respectively.
2. The Kronecker delta

The unit matrix is a 3×3 matrix of special character since it recovers the same matrix in multiplication. As a tensor it is of rank 2 and is denoted the unit tensor, or the **Kronecker delta**, written as $\delta_{ij}$. The matrix form is

$$\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \quad (15)$$

which in index notation is written

$$\delta_{ij} = \begin{cases} 
1; & \text{if } i = j \\
0; & \text{if } i \neq j
\end{cases} \quad (16)$$

The Kronecker delta is isotropic, i.e. it is the same in all coordinate systems. It is used in a variety of situations; one is the formation of general isotropic tensors, another is the tensor notation for the eigenvalue relation $Bx = \lambda x$ which is

$$(B_{ij} - \lambda \delta_{ij})x_j = 0 \quad (17)$$

The Kronecker delta is also quite useful in reducing expressions. As an example we take the expression $U_{ik}\delta_{ij}$. Here, summation is in index $i$. When this is performed, non-zero contributions appear only if $i = j$. Thus, the expression reduces to $U_{jk}$.
3. The permutation symbol and the vector product

The vector product is written with index notation as

\[
\mathbf{b} \times \mathbf{c} = \varepsilon_{ijk} \mathbf{b}_j \mathbf{c}_k
\]  

(18)

where \( \varepsilon_{ijk} \) is the permutation symbol having the properties

\[
\varepsilon_{ijk} = \begin{cases} 
1; & \text{if } ijk \text{ is } 123, 231 \text{ or } 312 \\
-1; & \text{if } ijk \text{ is } 132, 321 \text{ or } 213 \\
0; & \text{if two or more indices are equal}
\end{cases}
\]  

(19)

(19) may also be expressed as: \( \varepsilon_{ijk} = +1 \) if all indices are different and an even permutation of 123, \( \varepsilon_{ijk} = -1 \) if all indices are different and an odd permutation of 123 and \( \varepsilon_{ijk} = 0 \) if any two indices are equal. The permutation property thus result in \( \varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} \) and \( \varepsilon_{ijk} = -\varepsilon_{ikj} \). Finally, \( \varepsilon_{ijk} \) is also denoted the Levi-Cevita epsilon.
4. Isotropic tensors

Tensors with components independent of the coordinate system are denoted isotropic. Scalars have this property and are thus tensors (of rank 0). Thus, all tensors of rank 0 are isotropic. Vector components look different in different coordinate systems and can thus not be isotropic.

The Kronecker delta, $\delta_{ij}$, is an isotropic tensor of rank 2 and all isotropic tensors of rank 2 can be written $b \cdot \delta_{ij}$, where $b$ is a scalar.

The permutation symbol, $\varepsilon_{ijk}$, is an isotropic tensor of rank 3 and all isotropic tensors of rank 3 can be written $b \cdot \varepsilon_{ijk}$, where $b$ is a scalar.

If we multiply two isotropic tensors of rank 2 with different indices we obtain an isotropic tensor of rank 4. The most general form of such a tensor is

$$C_{ijkl} = b \cdot \delta_{ij} \delta_{kl} + c \cdot \delta_{ik} \delta_{jl} + d \cdot \delta_{il} \delta_{jk}$$  \hspace{1cm} (20)

where $b$, $c$ and $d$ are scalars. It may be shown that no other combination gives new information, but (20) can be rearranged to give

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) + \beta \left( \delta_{il} \delta_{jk} - \delta_{il} \delta_{jk} \right)$$  \hspace{1cm} (21)

where $\lambda$, $\mu$ and $\beta$ readily can be expressed in terms of $b$, $c$ and $d$. The specific choice of greek symbols here is arbitrary but conforms to the practice developed in solid mechanics.
5. The $\varepsilon$–$\delta$ relationship

There is a most useful relationship between the Kronecker delta and the permutation symbol,

$$\varepsilon_{klm} \varepsilon_{klm} = \varepsilon_{ijk} \varepsilon_{ilm} = \varepsilon_{ijk} \varepsilon_{klm} = \delta_{ij} \delta_{jm} - \delta_{im} \delta_{jl}$$

which is denoted the $\varepsilon$-$\delta$ relationship. Using this relationship, it is convenient to put the dummy index in the first or the last positions.

A motivation for (22) runs as follows: $\varepsilon_{ijk}$ is an isotropic tensor and the product of two isotropic tensors is also isotropic. Contracting an isotropic tensor gives another isotropic tensor. In particular, $\varepsilon_{klm} \varepsilon_{klm}$ is isotropic of rank 4. Referring to (20), we can write

$$\varepsilon_{klm} \varepsilon_{klm} = \alpha \delta_{ij} \delta_{jm} + \beta \delta_{im} \delta_{jl} + \gamma \delta_{im} \delta_{jl}$$

where $\alpha$, $\beta$ and $\gamma$ are to be determined. $\alpha$ is obviously zero since $\varepsilon_{ijk}$ is zero if $i$ and $j$ are equal but then $\delta_{ij}$ is unity. By examining the other combinations it may be shown that (22) is valid. The $\varepsilon$-$\delta$ relationship is most useful when simplifying complicated vector relationships.
6. Vector analysis and tensors

6.1 Grad, div and rot

The operator $\nabla$, (‘del’ or ‘nabla’) can be written as the vector

$$\nabla = \left( \begin{array}{c} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{array} \right)$$

(24)

Writing this in tensor form we have for the i-th component

$$[\nabla]_i = \frac{\partial}{\partial x_i}$$

(25)

where we use $[,]_i$ to select the i-th component.

The gradient of a scalar is written

$$[\text{grad}(c)] = [\nabla c] = \frac{\partial c}{\partial x_i} = c_i$$

(26)

where we have used the ‘comma notation’ as a compact writing for derivatives.

The divergence of a vector is written as

$$\text{div} \mathbf{b} = \nabla \cdot \mathbf{b} = \frac{\partial b_1}{\partial x_1} + \frac{\partial b_2}{\partial x_2} + \frac{\partial b_3}{\partial x_3} = \frac{\partial b_j}{\partial x_j} = b_{,j}$$

(27)

The curl of a vector is given by

$$[\text{curl}(\mathbf{b})]_i = [\nabla \times \mathbf{b}]_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} (b_k) = \varepsilon_{ijk} b_{,k}$$

(28)

6.2 Gauss’ theorem

Gauss’ theorem is a way of converting surface integrals to volume integrals, or vice versa. We assume this theorem to be known and show how it is written in tensor form. Starting with its vector form, Gauss’ theorem reads
\begin{equation}
\int_{S} \mathbf{b} \cdot \hat{n} \, dS = \int_{V} \text{div} \, \mathbf{b} \, dV
\tag{29}
\end{equation}

where the divergence is given above. In tensor form (29) becomes
\begin{equation}
\int_{S} b_{ki} n_{i} \, dS = \int_{V} \frac{\partial b_{j}}{\partial x_{j}} \, dV = \int_{V} b_{ji} \, dV
\tag{30}
\end{equation}

This can be generalized for a tensor of arbitrary rank to
\begin{equation}
\int_{S} b_{ki..} n_{i} \, dS = \int_{V} b_{ki..} \, dV
\tag{31}
\end{equation}

It is noted that the index for the normal vector in the surface integral is the same as for the derivative in the volume integral.

### 6.3 Vector formulas

Index notation is a most convenient tool when rewriting complicated vector relationships. As an example we take \( \mathbf{u} \times (\nabla \times \mathbf{u}) \) and note that both the cross product and the curl can be written using the permutation symbol. Then we can use the \( \varepsilon-\delta \) relationship. The steps are thus

\[
\varepsilon_{ijk} u_{j} = \epsilon_{jik} u_{i} = \delta_{ij} \delta_{km} u_{m,l} = \delta_{im} \delta_{jl} u_{j} u_{m,l} - \delta_{im} \delta_{jl} u_{j} u_{m,l} = u_{j} u_{j,-} - u_{j} u_{i,j}
\]

Noting that \( u_{j} u_{j,-} = \frac{1}{2} (u_{j} u_{j}), \) the first term is the i-th component of \( \nabla \left( \| \mathbf{u} \|^{2} / 2 \right) \). The other is identified as \( (\mathbf{u} \cdot \nabla) \) operating on \( u_{i} \). Thus, the given expression may be rewritten as
\begin{equation}
\mathbf{u} \times (\nabla \times \mathbf{u}) = \nabla \left( \| \mathbf{u} \|^{2} / 2 \right) - (\mathbf{u} \cdot \nabla) \mathbf{u}
\tag{33}
\end{equation}
Appendix A: Definition of a tensor

We have indicated that a tensor is defined according to how its components are transformed between coordinate systems. In order to define this in detail, we must first consider how coordinates are related between different systems. Consider figure A1 where two (orthogonal) coordinate systems are shown rotated relative to each other. Without loss of generality the origin of the systems coincide for all times (no translation).

\[ \mathbf{r} = x_i \mathbf{e}_i = x'_i \mathbf{e}'_i \]  

(A1)

If we multiply this expression by $\mathbf{e}'_j$, noting that $\mathbf{e}'_i \cdot \mathbf{e}'_j = \delta_{ij}$, we obtain

\[ x'_j = (\mathbf{e}'_j \cdot \mathbf{e}_i) x_i \]  

(A2)

The dot product is equal to the cosine of the angle between the two base vectors and is thus called the directional cosine. Obviously, this is the crucial parameter when translating between two coordinate systems and we therefore introduce the notation

\[ a_{ji} \equiv \mathbf{e}'_j \cdot \mathbf{e}_i \]  

(A3)

$a_{ji}$ is generally denoted the transformation coefficient.

(A2) can then be written as

\[ x'_j = a_{ji} x_i \]  

(A4)

The inverse of this expression is obtained by multiplying (A1) by $\mathbf{e}_j$. This gives
\[ x_j = (\xi_j \cdot \xi')x'_i = a_{ij}x'_i \quad \text{(A5)} \]

**A2. Properties of the transformation coefficients**

Combining (A4) and (A5) we obtain
\[ x_j = a_{ij}x'_i = a_{ij}a_{ik}x_k \quad \text{(A6)} \]
From (A6) we draw the conclusion that
\[ a_{ij}a_{ik} = \delta_{jk} \quad \text{(A7)} \]
Similarly, it is shown that
\[ a_{ji}a_{ki} = \delta_{jk} \quad \text{(A8)} \]

**A3. Transformation of tensors of rank 1**

From the transformation of coordinates it follows that the derivations also apply to any other vector. Thus, we conclude that tensor of rank 1 (a vector) is transformed according to the rule
\[ b'_i = a_{ij}b_j \quad \text{(A9)} \]

**A4. Transformation of tensors of rank 2**

To illustrate how higher rank tensors transform we consider the stress tensor \( \sigma_{ij} \). The Cauchy relationship between the stress vector at a point (the traction) and the stress tensor is given by \( t_i = \sigma_{ji}n_j \). This applies also in the transformed system. Thus,
\[ t'_i = \sigma'_{ji}n'_j \quad \text{(A10)} \]
Here, we use that \( t'_i = a_{ik}t_k \) and \( n'_j = a_{ji}n_j \). Then, (A10) becomes
\[ t'_i = a_{ik}t_k = \sigma'_{ji}a_{ji}n_j \quad \text{(A11)} \]
Multiplying this expression by \( a_{im} \) and using (A7) we obtain
\[ t_m = \sigma'_{ji}a_{ji}a_{im}n_i = \sigma_{jm}n_j \quad \text{(A12)} \]
Changing the dummy index in the right hand side to \( l \) and noting that the direction \( n \) is arbitrary gives
\[ \sigma'_{j} a_{jm} a_{m} = \sigma_{ln} \]  
\[(A13)\]

This can be inverted by multiplying by \(a_{kl} a_{lm}\) and using (A7) which gives
\[ \sigma'_{kn} = a_{kl} a_{mn} \sigma_{lm} \]  
\[(A14)\]

The relation (A14) is the general transformation rule for 2\textsuperscript{nd} rank tensors. Thus,
\[ b'_{ij} = a_{im} a_{jm} b_{mn} \]  
\[(A15)\]

**A5. Transformation of tensors of general rank**

Following the pattern set up by the previous examples we define the transformation rule for a tensor of general rank as
\[ b'_{ijkl...} = a_{im} a_{jn} a_{ko} a_{lp} ... b_{mnop...} \]  
\[(A16)\]

Note the pattern: The first index in the transformation coefficients belongs to the transformed tensor. The second index belongs to the original tensor.

**A6. Connection to matrix notation for 2nd rank tensors**

A tensor of rank 2 can also be identified as a matrix. This means that matrix relations have tensor counterparts. As an example, (A15) can be seen as a matrix multiplication
\[ b'_{ij} = a_{ik} a_{jl} b_{kl} = a_{ik} b_{kl} a_{jl} \]  
\[(A17)\]

Here, \(a_{ik} b_{kl} = (AB)_{ij}\) and \(a_{jl} = (A^T)_{ij}\). Thus, (A17) is equivalent to the matrix multiplication
\[ B' = A B A^T \]  
\[(A18)\]

**A7. Formal definition of a tensor**

Based on the previous derivations, we define a tensor as a quantity that transforms between coordinate systems as
\[ b'_{ijkl...} = a_{im} a_{jn} a_{ko} a_{lp} ... b_{mnop...} \]  
\[(A19)\]

The formal definition can be used to decide whether a quantity is a tensor or not. E.g. products, sums, differences and combinations thereof are tensors.
Appendix B: Invariants of a matrix

The eigenvalue relation for a matrix, $Bx = \lambda x$, can be written in tensor form as

$$ (B_{ij} - \lambda \delta_{ij})x_j = 0. \quad (B1) $$

For a non-trivial solution of (B.1), the determinant

$$ \det(B_{ij} - \lambda \delta_{ij}) = 0. \quad (B2) $$

must be zero. Calculation of the determinant gives

$$ (B_{11} - \lambda)(B_{22} - \lambda)(B_{33} - \lambda) - (B_{11} - \lambda)B_{32}B_{23} + B_{12}B_{23}B_{31} - B_{12}(B_{33} - \lambda)B_{21} + B_{13}B_{12}B_{33} - (B_{22} - \lambda)B_{13}B_{31} = $$

$$ -\lambda^3 + \lambda^2(B_{11} + B_{22} + B_{33}) $$

$$ - \lambda(B_{11}B_{22} + B_{11}B_{33} + B_{22}B_{33} - B_{12}B_{21} - B_{13}B_{31} - B_{23}B_{32}) $$

$$ + B_{11}B_{22}B_{33} + B_{12}B_{23}B_{31} + B_{13}B_{21}B_{32} - B_{12}B_{23}B_{31} - B_{13}B_{21}B_{32} - B_{31}B_{32} $$

This may be written

$$ \lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \quad (B2) $$

where

$$ I_1 = B_{11} + B_{22} + B_{33} = B_1, \quad (B4) $$

$$ I_2 = B_{11}B_{22} + B_{11}B_{33} + B_{22}B_{33} - B_{12}B_{21} - B_{13}B_{31} - B_{23}B_{32} $$

$$ = B_{11}B_{22} + B_{11}B_{33} + B_{22}B_{33} + \frac{1}{2}(B_{11}^2 + B_{22}^2 + B_{33}^2) $$

$$ - \frac{1}{2}(B_{11}^2 + B_{22}^2 + B_{33}^2) - B_{12}B_{21} - B_{13}B_{31} - B_{23}B_{32} $$

$$ = \frac{1}{2}(B_{11} + B_{22} + B_{33})^2 $$

$$ - \frac{1}{2}(B_{11}^2 + B_{22}^2 + B_{33}^2 + B_{12}B_{21} + B_{13}B_{31} + B_{23}B_{32} + B_{12}B_{23} + B_{13}B_{32} + B_{21}B_{13} + B_{23}B_{12} + B_{31}B_{32}) $$

$$ = \frac{1}{2}(B_{ij}B_{ji} - B_{ij}B_{ji}) \quad (B5) $$

and
\( I_3 = B_{11}B_{22}B_{33} + B_{12}B_{23}B_{31} + B_{13}B_{21}B_{32} - B_{32}B_{23}B_{11} - B_{12}B_{21}B_{33} - B_{13}B_{31}B_{22} \)

\[ = \det B \quad (B6) \]

The three scalars \( I_1, I_2 \) and \( I_3 \) are denoted the **invariants of \( B \)** since they are independent of the coordinate system. The invariant \( I_2 \) can be written in terms of sub-determinants as

\[
I_2 = \frac{1}{2} \left( B_{ij}B_{ji} - B_{ij}B_{ji} \right) = \\
\begin{vmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{vmatrix} + \\
\begin{vmatrix}
B_{22} & B_{23} \\
B_{32} & B_{33}
\end{vmatrix} + \\
\begin{vmatrix}
B_{11} & B_{13} \\
B_{31} & B_{33}
\end{vmatrix}
\quad (B7)
\]

**Summary:**

\[ I_1 = B_{ii} \]
\[ I_2 = (B_{ij}B_{ji} - B_{ij}B_{ji})/2 \]
\[ I_3 = \det B \]