Signal Processing & Fourier Analysis

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Short Course Outline

- Day 1
  - ◇ Introduction & History
  - ◇ Mathematical Preparation/Context
  - ◇ Fourier Series
  - ◇ Lunch Break
  - ◇ Lab work I
- Day 2
  - ◇ L^2 Theory
  - ◇ Fourier Transform
  - ◇ Discrete Fourier
  - ◇ Points in Space (a digression)
  - ◇ Applications
  - ◇ Lunch Break
  - ◇ Lab work II

Introduction & History

Course Material

Course material will be drawn from

- “Fourier Analysis and Its Applications” by Anders Vretblad, Springer.
- Some personal notes and perspectives

Fourier?

What do you think of when you hear the term “Fourier” ??

Fourier, the person

- Jean Baptiste Joseph Fourier 1768-1830
- French mathematician and physicist
- discovered “greenhouse effect”
- studied heat transfer
- “Theorie Analytique de la Chaleur” (1822)
- known for Fourier Series, Fourier Transform

Some Notation

- LaPlace Operator $\triangle = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
- partial with respect to time $u_t = \frac{\partial u}{\partial t}$
- second partial $u_{xy} = \frac{\partial^2 u}{\partial x \partial y}$
- denote 3-d space as $\Omega = [x \quad y \quad z]$

Well-posed Problems

A problem is said to be "well-posed" when all three conditions are met:

- there exists a solution to the problem
- there exists *only one* solution
- the solution is *stable* (small changes in equation parameters produce small changes in solution)

Some Important Historical Physical Equations

- The Wave Equation
- The Heat Equation
- The LaPlace Equation
- The Poisson Equation

We'll look at the first two more closely.

The Wave Equation

$$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad (x,t) \in \Omega \times T$$

- $u(t,x)$ is the displacement at time t point x
- c is a constant depending on properties of material
- describes vibrations in a homogeneous medium
- reversible process

On a string, we can write:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

One Dimensional Wave Equation

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

- Consider a sol'n in open half plane $t > 0$
- introduce new coordinates $\xi = x - ct$, and $\eta = x + ct$
- $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial \xi^2} + 2\dfrac{\partial^2 u}{\partial \xi \partial \eta} + \dfrac{\partial^2 u}{\partial \eta^2}$
- $\dfrac{\partial^2 u}{\partial t^2} = c^2 \left(\dfrac{\partial^2 u}{\partial \xi^2} - 2\dfrac{\partial^2 u}{\partial \xi \partial \eta} + \dfrac{\partial^2 u}{\partial \eta^2} \right)$

One Dimensional Wave Equation (cont.)

- Inserting these into equation yields:

$$c^2\, 4\, \frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \qquad \Leftrightarrow \qquad \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta}\right) = 0$$

- We see $\frac{\partial u}{\partial \eta}$ is only a function of η, say $\frac{\partial u}{\partial \eta} = h(\eta)$
- Let ϕ be antiderivative of h, then another integration yields $u = \phi(\eta) + \psi(n)$, where ϕ is a new arbitrary function.
- returning to original variables (x,t) we have found that

$$u(x,t) = \phi(x-ct) + \psi(x+ct)$$

- ϕ and ψ are more or less arbitrary functions of one variable.
- Note motion "to the left" and "to the right"

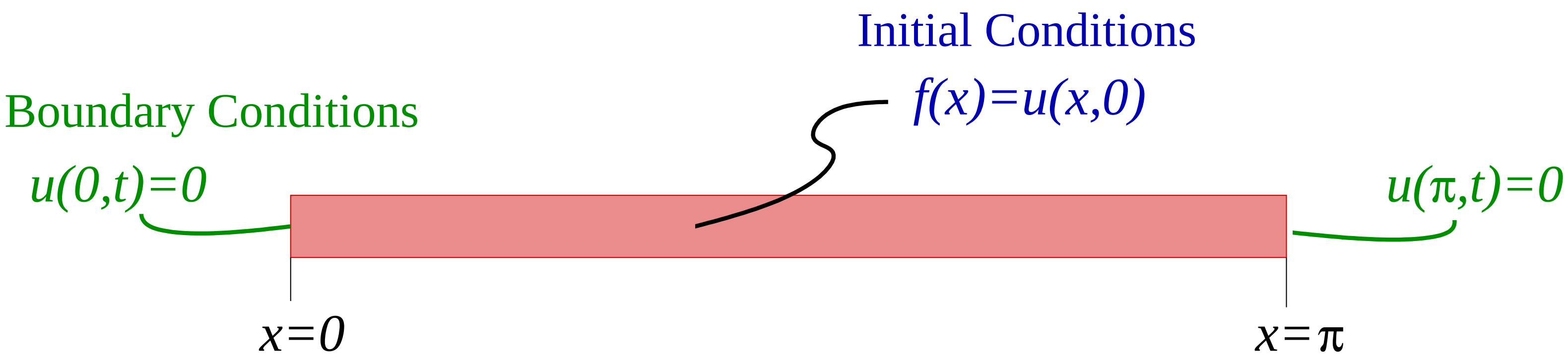
The Heat Equation / The Diffusion Equation

$$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{a^2}\frac{\partial u}{\partial t} \qquad (x,t) \in \Omega \times T$$

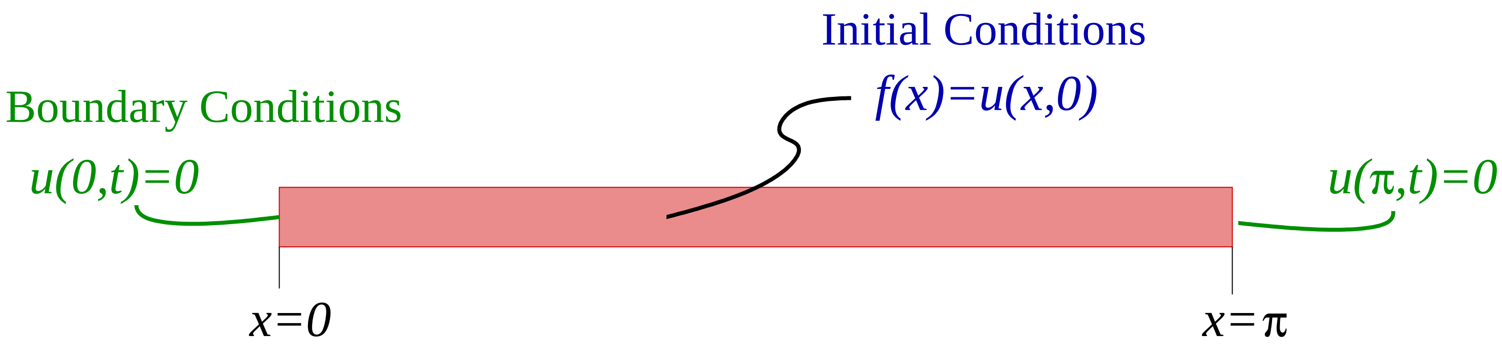
- $u(t,x)$ is the temperature at time t at point x
- describes the heat flow per unit time
- a is a constant depending on properties of material
- irreversible process

Fourier's Method (Theorie Analytique de la Chaleur-1822)

- Attempt to solve Heat Equation $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$
- assume rod of length π
- keep each end at temperature zero $$u(0,t) = u(\pi,t) = 0$$
- assume an initial temperature distribution within rod is $$f(x) = u(x,0)$$



Fourier's Method (cont.)



Consider:

- equation **(E)** $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad 0 < x < \pi, \qquad t > 0$$
- boundary cond. **(B)** $$u(0,t) = u(\pi,t) = 0 \qquad t > 0$$
- intial cond. **(I)** $$u(x,0) = f(x) \qquad 0 < x < \pi$$

These are linear, if any u and v meets these, then so does $\alpha u + \beta v$!

Fourier's Idea

- solve for partial problem consisting of just
  - $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ **(E)**
  - $u(0,t) = u(\pi,t) = 0$ **(B)**
- considered solutions of form $u(x,t) = X(x)T(t)$
  - $X(x)$ depends on just one variable
  - $T(t)$ depends on just one variable
- method is called "separation of variables"

Fourier's Idea (cont.)

- when considering solutions of form $u(x,t) = X(x)T(t)$ then **(E)**,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad \text{becomes} \qquad X''(x)T(t) = X(x)T'(t) \qquad 0 < x < \pi, t > 0$$

- rearrange as

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} \qquad 0 < x < \pi, \qquad t > 0$$

- has peculiar property that if we change $t \longrightarrow$ LHS is uneffected $\longrightarrow$ RHS is also uneffected. So this must be a **constant** (call this $-\lambda$)

- Similarly for changes of x

Fourier's Idea (cont.)

- So, we have now $$-\lambda = \frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)}$$
- or,

$$X''(x) + \lambda X(x) = 0 \qquad 0 < x < \pi$$

$$T'(t) + \lambda T(t) = 0 \qquad t > 0$$

- including **(B)** by $u(x,t) = X(x)T(t)$ yields:
  - ◇ $X(0)T(t) = X(\pi)T(t) = 0 \qquad t > 0$
  - ◇ if $X(0) \neq 0 \;\rightarrow\; T(t) = 0$ for $t > 0 \;\Rightarrow\; u(x,t) = 0$ (trivial sol'n)
  - ◇ to get interesting sol'n, we must demand $X(0) = X(\pi) = 0$

Fourier's Idea (cont.)

We then consider the boundary value problem of:

- $X''(x) + \lambda X(x) = 0 \qquad 0 < x < \pi$

  with $X(0) = X(\pi) = 0$

- must consider three cases of λ:
  - ◇ $\lambda < 0$
  - ◇ $\lambda = 0$
  - ◇ $\lambda > 0$

Case $\lambda < 0$

$$X''(x) + \lambda X(x) = 0, \qquad X(0) = X(\pi) = 0,$$

- can write $\lambda = -\alpha^2$, and assume $\alpha > 0$, yielding $\qquad X''(x) - \alpha^2 X(x) = 0$
- general sol'n is $\qquad X(x) = Ae^{\alpha x} + Be^{-\alpha x}$
- **(B)** becomes $$0 = X(0) = A + B$$ $$0 = X(\pi) = Ae^{\alpha\pi} + Be^{-\alpha\pi}$$
- homogenous linear system of equations, two equations, two unknowns
- this has **uninteresting**, unique sol'n $\qquad A = B = 0$
- then $X(x)$ is trivial (and uninteresting)

Case $\lambda = 0$

$$X''(x) + \lambda X(x) = 0, \qquad X(0) = X(\pi) = 0,$$

- differential equation reduces to $\qquad X''(x) = 0$
- this has solution $\qquad X(x) = Ax + B$
- again **(B)**, $X(0) = X(\pi) = 0 \qquad \Rightarrow \quad A = B = 0$
- $X(x)$ is again trivial (and uninteresting)

Case $\lambda > 0$

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(\pi) = 0,$$

- let $\lambda = \omega^2$, and assume $\omega > 0$, yielding $\quad X''(x) + \omega^2 X(x) = 0$
- general sol'n is $\quad X(x) = A\cos\omega x + B\sin\omega x$
- **(B)** of $X(0) = 0 \quad \Rightarrow A = 0$
- **(B)** of $X(\pi) = 0 \quad \Rightarrow X(x) = 0 = B\sin\omega\pi$
- if $B = 0 \rightarrow$ uninteresting
- but $B \neq 0$ becomes interesting!

  $B\sin\omega\pi = 0$ has sol'n whenever ω is pos. integer

Almost!

- the problem $X''(x) + \lambda X(x) = 0, \quad 0 < x < \pi, \quad X(0) = X(\pi) = 0$
- has non-trivial sol'n exactly for $\lambda = n^2$ where n is pos. integer
- this sol'n has form of $X(x) = X_n(x) = B_n \sin nx$

  where B_n is a constant
- For these values of λ, let's also solve
$$T'(t) + \lambda T(t) = 0$$
$$T'(t) = -n^2 T(t)$$
- This has general sol'n $T(t) = T_n(t) = C_n e^{-n^2 t}$

Finally!

- We have
$$X(x) = B_n \sin nx, \quad n \text{ is pos. int.}$$
$$T(t) = T_n(t) = C_n e^{-n^2 t}$$

- Combining these two partial results, we have

$$\begin{aligned} u(x,t) &= X(x)\,T(t) = B_n \sin nx \; C_n e^{-n^2 t} \\ &= \underbrace{B_n C_n}_{b_n} \sin nx \, e^{-n^2 t} = b_n \, e^{-n^2 t} \sin nx \end{aligned}$$

- By **linearity**, all sums of such expressions are also solutions:

$$u(x,t) = \sum_{n=1}^{N} b_n \, e^{-n^2 t} \sin nx$$

Summary of Fourier's Result

- for **(I)**, initial conditions, of form

$$f(x) = u(x,0) = \sum_{n=1}^{N} b_n \sin nx \qquad 0 < x < \pi$$

- for **(B)**, boundary conditions $\qquad u(0,t) = u(0,\pi) = 0$

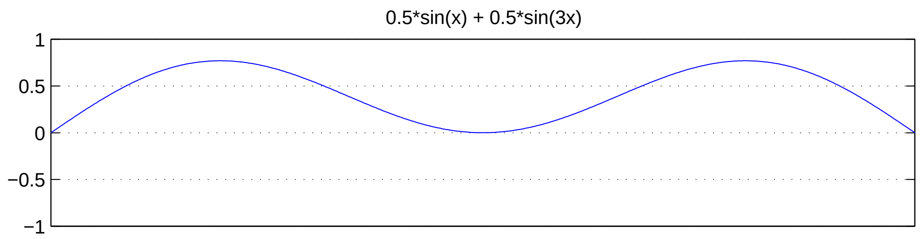
- one dimensional heat equation $\qquad c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$

- has solutions of the form:

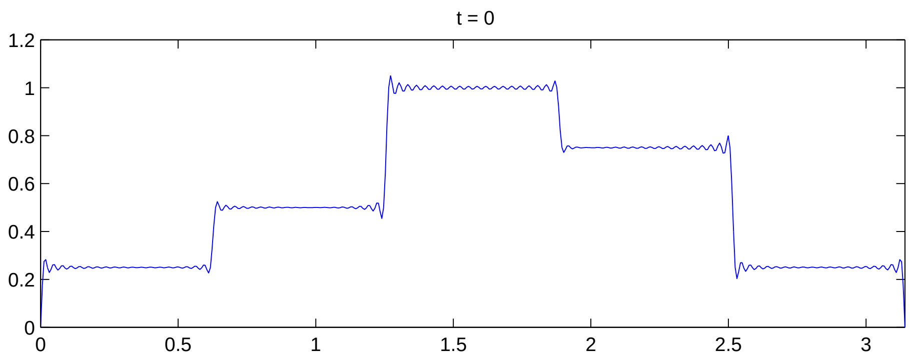
$$u(x,t) = \sum_{n=1}^{N} b_n \, e^{-n^2 t} \sin nx \qquad 0 < x < \pi, \quad t > 0$$

Numerical Experiment of Fourier's Result

- We use MATLAB with Fourier's equation of solution to visualize two cases:
  - ◇ temperature in a bar with $u(x,0) = f(x) = \frac{1}{2}\sin x + \frac{1}{2}\sin 3x$



  - ◇ temperature in a bar with $u(x,0) = f(x)$ with many terms



Questions on Fourier's Result

- Can we permit $N \longrightarrow \infty$?
- Is it possible to approximate an *arbitrary* function $f(x)$ using sums of sinusoids?

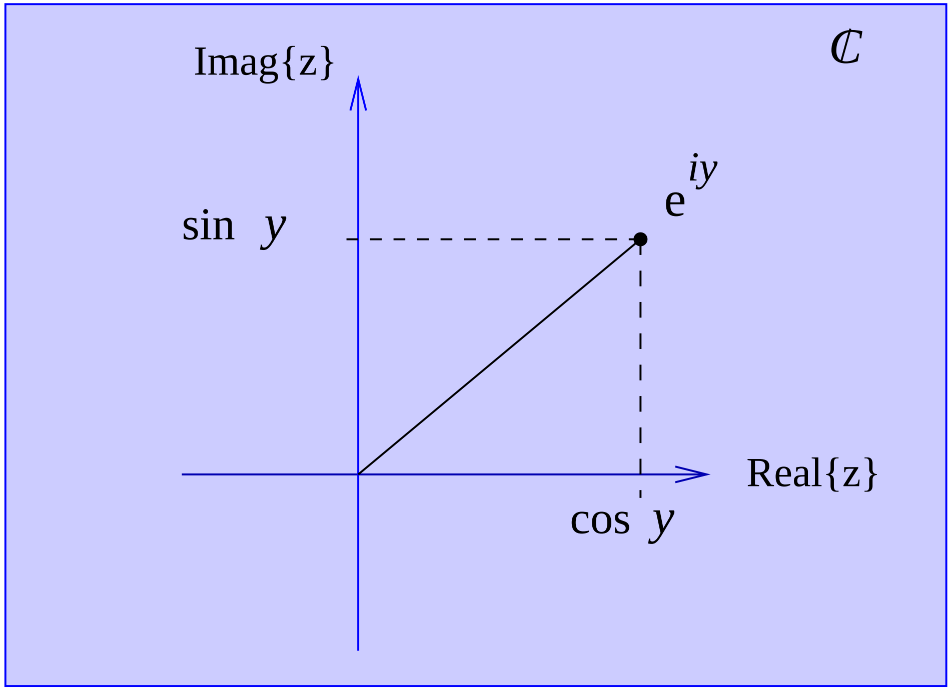
Remarks

- Initially, usefulness of Fourier's results were met with some scepticism
- The extension of $f(x)$ to arbitrary functions was considered controversial
- Ideas on convergence of function needed to be more fully developed to assess situation
- The answer to these (and other questions) were a long time coming (ending in 1960's (?))

Some Mathematical & Notational Preparation

Euler's Formula

- Euler's Formula $$e^{iy} = \cos y + i \sin y$$



Positive summation kernels (or distributions)

- let $I = (-a, a)$ be an interval (finite or infinite)
- suppose $\{K_n(s)\}_{n=1}^{\infty}$ has properties

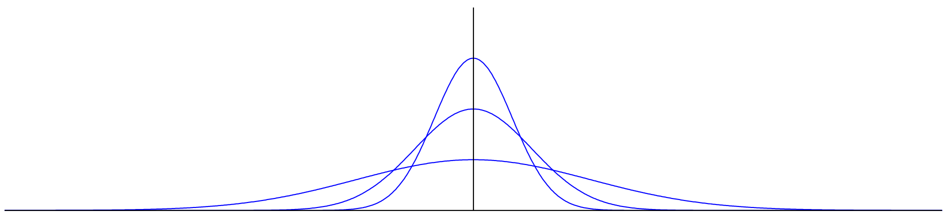
(1) $K_n(s) \geq 0, \quad \forall s \in I$

(2) $\displaystyle\int_{-a}^{a} K_n(s)ds = 1$

(3) if $\delta > 0$, then $\displaystyle\lim_{n\to\infty} \int_{\delta<|s|<a} K_n(s)ds = 0$

- If $f : I \to C$ is integrable and bounded on I and continuous for $s = 0$,

we then have $\displaystyle\lim_{n\to\infty} \int_{-a}^{a} K_n(s) f(s)ds = f(0)$



Generalization of a Function

- Dirac Distribution (or "Delta Function"):

(1) $\delta(t) \geq 0, \quad -\infty < t < \infty$

(2) $\delta(t) = 0, \quad t \neq 0$

(3) $\int_{-\infty}^{\infty} \delta(t) dt = 1$

- Properties of Dirac Distribution:

◇ $\int_{-\infty}^{\infty} \delta(t)\phi(t) dt = \phi(0)$

◇ $\int_{-\infty}^{\infty} \delta(t-\tau)\phi(t) dt = \phi(\tau)$

Review of Some Linear System Theory

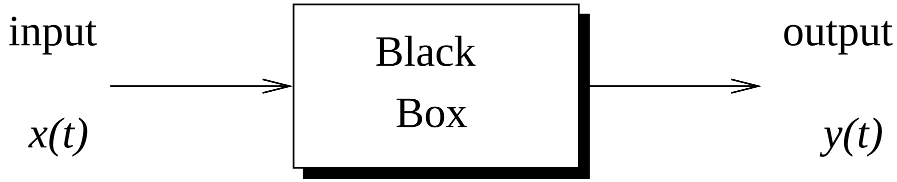
Laplace Transform

- Pierre Simon de Laplace, *Théorie analytique des probabilités* (1812)
- His methods "baffled his contemporaries".
- Defined as:

$$\tilde{f}(s) = \int_0^{\infty} f(t) e^{-st} dt \; = \; \mathcal{L}[f(t)]$$

- Useful tool for solving linear differential equations with initial value conditions
- It is linearity $\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)]$

Convolution



- Assume system has four properties:

i) *Linearity*. If $x_1(t)$ produces $y_1(t)$, and

$x_2(t)$ produces $y_2(t)$,

then

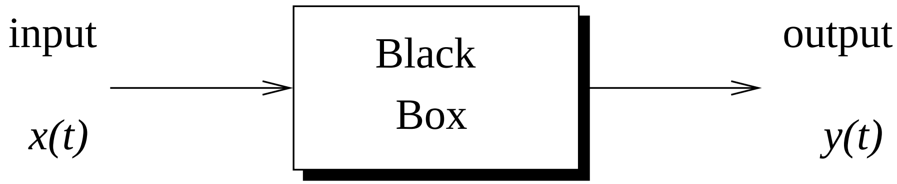
$$\alpha x_1(t) + \beta x_2(t) \quad \text{produces} \quad \alpha y_1(t) + \beta y_2(t)$$

ii) *Time Invariance* If $x(t)$ produces $y(t)$,

then

$x(t-\tau)$ produces $y(t-\tau)$,

Convolution (cont.)



iii) *Continuity.* continuous “small” changes in input $x(t)$, produce continuous “small” changes in output $y(t)$

iv) *Causality.* Output y at time t does not depend on input x at a time later than t.

Convolution and Impulse Response

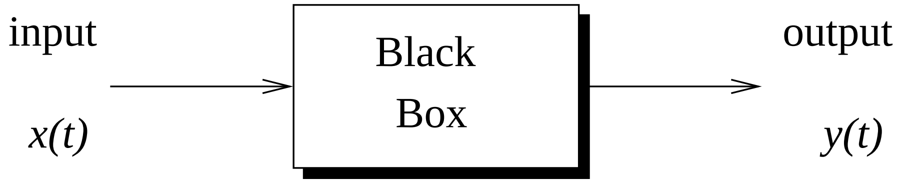
With these four conditions met we have:

- there exists a $g(t)$ such that

$$y(t) = \int_0^t x(\lambda)g(t-\lambda)d\lambda = \int_0^t x(t-\lambda)g(\lambda)d\lambda$$

- $g(t)$ contains all the information about the system
  - ◇ $g(t)$ is known as the system's ”impulse response”
  - ◇ integral is known as the ”convolution integral”

Linear Systems and Sinusoids (Complex Exponentials)



- An interesting property of linear systems is that

  "sinusoidal input $\longrightarrow$ sinusoidal output"

  - ◇ frequency of output is same as frequency of input
  - ◇ only phase, and amplitude may change,
  - ◇ often a measure of how linear (or non-linear) a system is (for example THD)

- "complex exponentials are eigenfunctions of linear systems"

$\mathcal{Z}$-Transform

- Consider the sequence $\{a_n\}_{n=0}^{\infty}$

- form the summation, $$A(z) = \sum_{n=0}^{\infty} a_n z^{-1}$$ where $z \in \mathcal{C}$

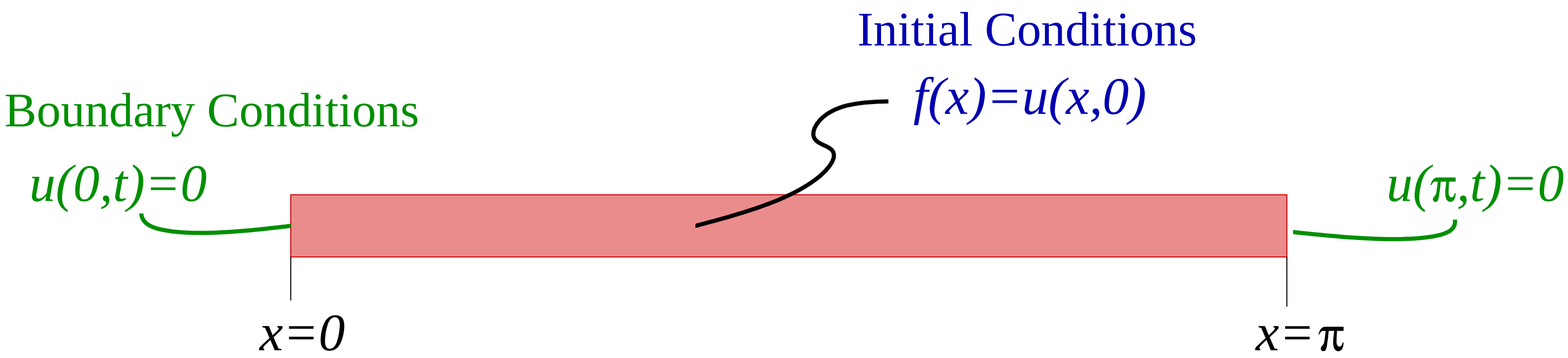
- for those z for which the sum converges we call $A(z)$ the " $\mathcal{Z}$-Transform of a_n".

- often considered analogous to the Laplace Transform for discrete sequences

Fourier Series

Fourier Series

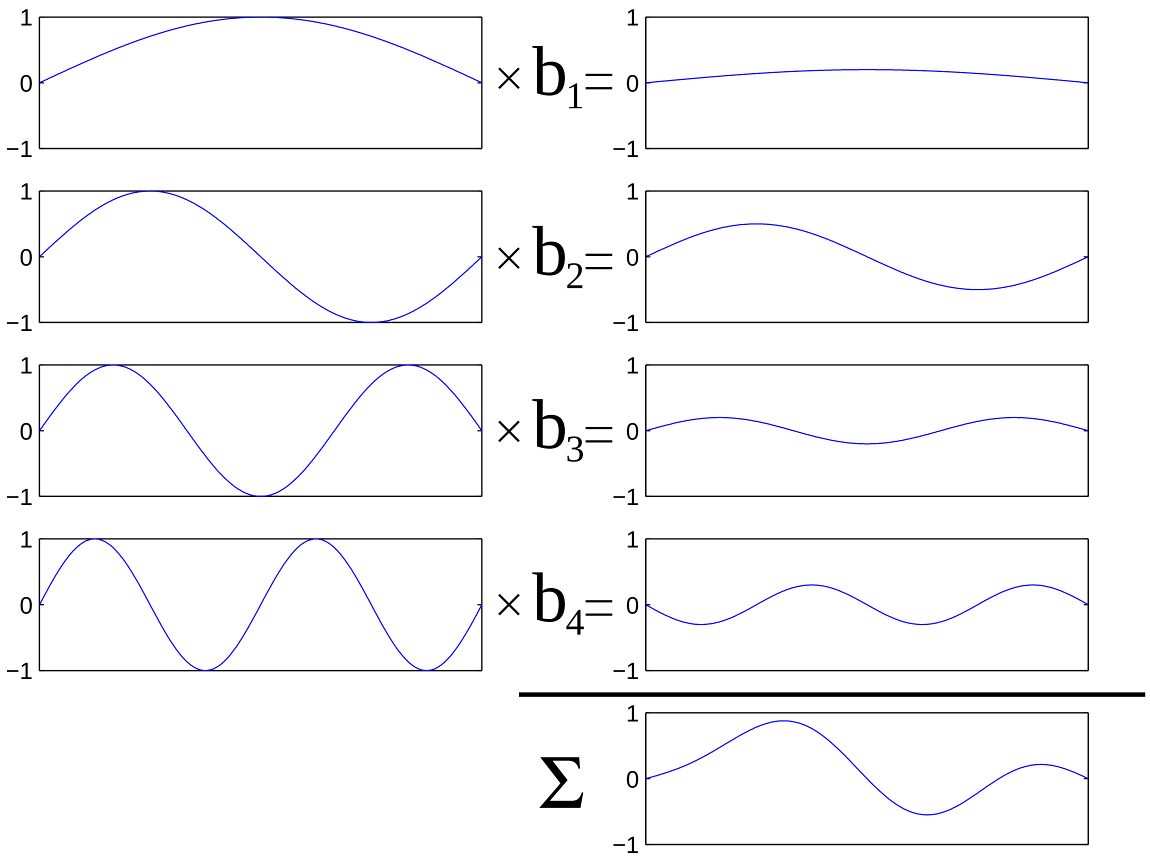
We return to Fourier's solutions of the heat problem



- Found solutions of form $$u(x,t) = b_n\, e^{-n^2 t}\, \sin nx$$
- for initial conditions $$f(x) = u(x,0) = \sum_{n=1}^{N} b_n \sin nx \quad 0 < x < \pi$$

How general is this solution?

What functions can be created on $[0, \pi]$ by $\sum_{n=1}^{N} b_n \sin nx$?



More General Form

- Let's shift from $f(t) = \sum_{n=1}^{N} b_n \sin nt$ on $[0, \pi]$ ...
- ... to a more general form of

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int} = \sum_{n=-\infty}^{\infty} c_n (\cos nt + i \sin nt)$$ and extend over all t.

- Can ask similary question ...

What are the $f(t)$ that can now be constructed?

- a bit of reflection yields that $f(t)$ will be *periodic* with period $T = 2\pi$

$$f(t) = f(t+T)$$

- also note $\int_T = \int_{-\pi}^{\pi} = \int_0^{2\pi} = \int_a^{a+2\pi}$ for any $a \in \mathcal{R}$

Relations

- Suppose $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$ and also assume $\sum_{n=-\infty}^{\infty} |c_n| < \infty$
- Consider
$$\int_{-\pi}^{\pi} f(t)e^{-imt}dt = \int_{-\pi}^{\pi} \left(\sum_{n=-\infty}^{\infty} c_n e^{int} \right) e^{-imt}dt$$
$$= \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} c_n e^{-i(n-m)t}dt$$
$$= \sum_{n=-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{-i(n-m)t}dt$$
- Look at
$$\int_{-\pi}^{\pi} e^{ikt}dt = \begin{cases} 2\pi & k = 0 \\ 0 & k \neq 0 \end{cases}$$
- So, we have
$$\int_{-\pi}^{\pi} f(t)\, e^{-imt}dt \;=\; 2\pi\, c_m$$

Obtaining c_m from $f(t)$

- Interestingly, we now see that if $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$ ...
- ... then we could find the c_m from the integral

$$c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-imt} dt$$

Fourier Series, Definition

Def'n 4.1 Let f be a function with period 2π that is absolutely Riemann integrable over a period.

Define the numbers c_n, with $n \in \mathbb{Z}$ by

$$c_n = \frac{1}{2\pi} \int_T f(t)\, e^{-int} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-int} dt.$$

- c_n are called "Fourier coefficients of f"
- The "Fourier Series of f" is the series $\sum_{n \in \mathbb{Z}} c_n e^{int}$

What about convergence?

- We've made no statement about if series converges.
- Even if the series converges, we've made no statement about what the series converges to.
- Let's play with this a bit in MATLAB
  - ◇ construct a vector t, and define some function $f(t)$
  - ◇ calculate c_n for $f(t)$
  - ◇ look at series for an increasing number of terms

Convergence of Fourier Series

- It certainly looks like the series $\sum_{n=-\infty}^{\infty} c_n e^{int}$ is approximating (or converging to) $f(t)$ as # of terms increases.
  - ◇ What do we mean by convergence here?
  - ◇ What do we mean by approximation here?

- Much of Chapter 4 is dedicated to the details of the convergence question.

End Result, Theorem 4.2

If

- f is a continuous on T,
- and its Fourier coefficients c_n are such that $\sum_{n=-\infty}^{\infty} |c_n|$ is convergent,

then the Fourier Series is

- convergent with sum $= f(t)$ for all $t \in T$,
- and the convergence is uniform on T

Quite a remarkable path!

- Our original motivation was to see how general Fourier's heat equation solution was.
- Our investigation leads to the result that an extremely large class of continuous, 2π periodic function could be constructed from complex exponentials ($\sum_{n \in \mathbb{Z}} c_n e^{int}$)
- We could say that we "build up" an arbitrary, continuous, 2π periodic function by taking a "weighted sum" of e^{int}.
- This leads to the ideas of "bases" and "basis functions"

We investigate this tomorrow!

Day 2

Yesterday

- We could say that we "build up" an arbitrary, continuous, 2π periodic function by taking a "weighted sum" of e^{int}.
- This leads to the ideas of "bases" and "basis functions"
- Why is this Fourier idea so big?
- What's the fascination with these complex exponential functions e^{int}?

Why are e^{int} so important?

- Partial differential equations often have (damped) complex exponentials as their solution.
- Mechanical systems often *vibrate* or *resonate* with periodic characteristics.
- Complex exponentials are *eigenfunctions* of linear systems!

To explore more, we need to develop ideas, known as L^2 Theory

Inner Product

Def'n 5.1 Let V be a complex vector space. An inner product on V is a complex-valued function $\langle u, v\rangle$ of u and $v \in V$ having the following properites:

- $\langle u, v\rangle = \overline{\langle v, u\rangle}$
- $\langle \alpha u + \beta v, w\rangle = \alpha \langle u, w\rangle + \beta \langle v, w\rangle$
- $\langle u, u\rangle \geq 0$
- $\langle u, u\rangle = 0 \quad \Rightarrow \quad u = 0$

Examples of an Inner Product

Example 5.3. Let $C(a,b)$ be the set of continuous, complex-valued functions defined on the compact interval $[a,b]$ and set

$$\langle f,g\rangle = \int_a^b f(x)\overline{g(x)}dx.$$

This is an inner product.

Another Example Let $u,v \in \mathcal{C}^N$ (say, N-dimensional column vectors). Then,

$$\langle u,v\rangle = u^T\overline{v} = [u_1 \quad u_2\ldots u_N]\begin{bmatrix}\overline{v_1}\\ \overline{v_2}\\ \vdots\\ \overline{v_N}\end{bmatrix} = \sum_{k=1}^{N} u_k\overline{v_k}$$

Properties of Inner Product

Inner productions satisfy:

- $|\langle u, v\rangle| \leq \|u\| \ \|v\|$ (Cauchy-Schwarz Inequality)
- $|u+v| \leq \|u\| + \|v\|$ (Triangle Inequality)

In a sense,

- inner products give us the idea of “distance”
- inner products define a “geometry”

Orthogonal Projections

Let $\{\phi_k\}_{k=1}^{N}$ be an orthonormal set in the space V, and let u be an arbitrary vector in V.

The orthogonal projection of u on to the subspace of V spanned by $\{\phi_k\}_{k=1}^{N}$ is the vector,

$$
\begin{aligned}
P_N(u) &= \langle u, \phi_1 \rangle\, \phi_1 + \langle u, \phi_2 \rangle\, \phi_2 + \ldots \langle u, \phi_N \rangle\, \phi_N \\
&= \sum_{k=1}^{N} \underbrace{\langle u, \phi_k \rangle}_{\text{coeffs}} \underbrace{\phi_k}_{\text{basis vector}}
\end{aligned}
$$

Useful Theorem

Thm 5.2 If $\phi_1, \phi_2, \ldots, \phi_N$ is an orthonormal basis in a N-dimensional inner product space V. Then every $u \in V$ can be written as

$$u = \sum_{j=1}^{N} \langle u, \phi_j \rangle \, \phi_j$$

and furthermore one has

$$\|u\|^2 = \sum_{j=1}^{N} |\langle u, \phi_j \rangle|^2$$

Crowning the Fourier System!

The two orthogonal systems

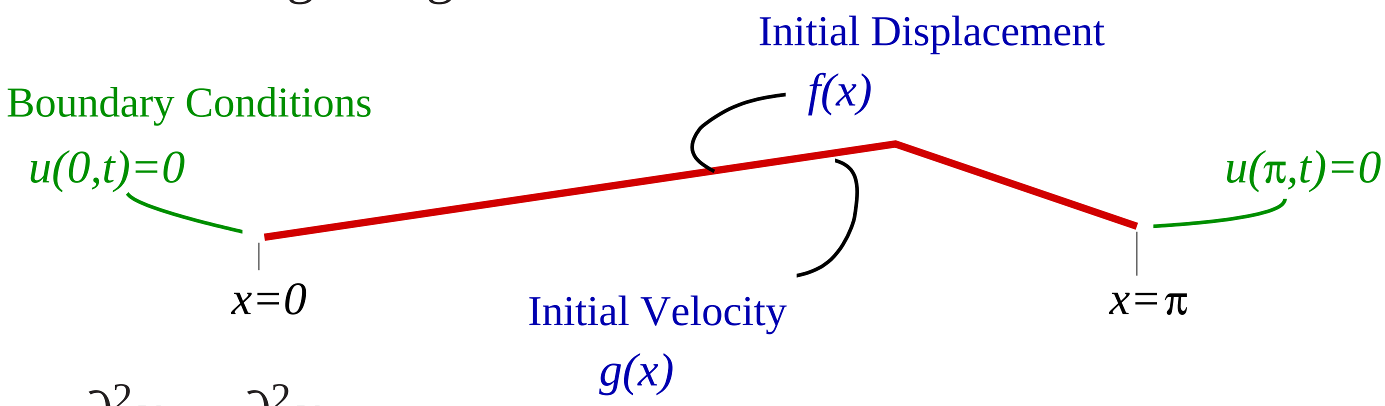
- $\{e^{int}\}_{n\in\mathbb{Z}}$
- $\{\cos nt, n \geq 0; \sin nt, n \geq 1\}_n$

are each complete in $L^2(T)$.

Loosely, in other words, any square integrable, T-periodic function can be constructed from these basis elements!

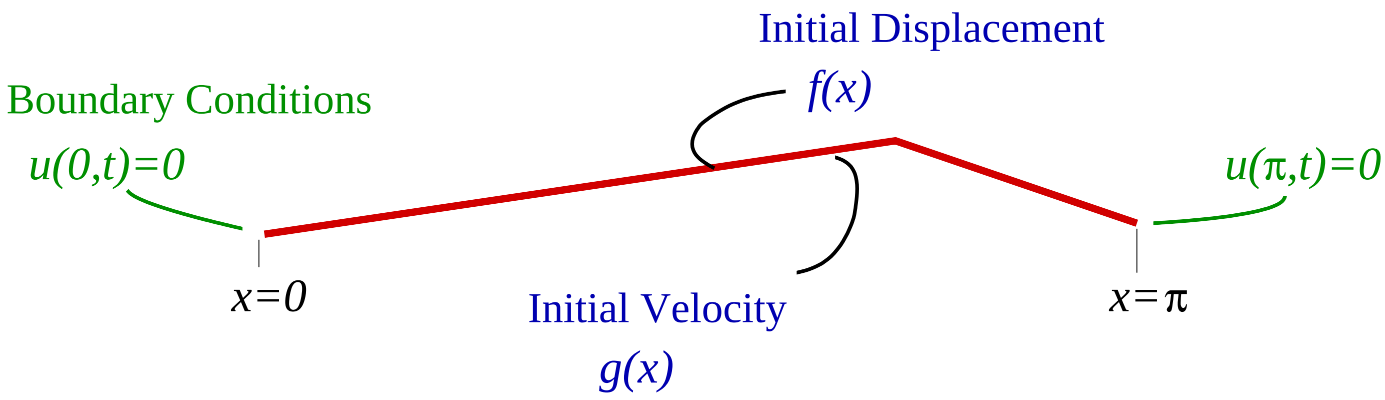
Back to the 1-D Wave equation

Consider a vibrating string



- **(E)** $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$ Equation
- **(B)** $u(0,t) = u(\pi,t) = 0$ Boundary Conditions
- **(I$_1$)** $u(x,0) = f(x)$ Inital (Displacement) Conditions
- **(I$_2$)** $\dfrac{\partial u(x,0)}{\partial t} = g(x)$ Inital (Velocity) Conditions

1-D Wave equation, Vibrating String



The same separation of variables techniques yields solution:

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \sin nx$$

- notice we have oscillations in time
- notice dependence also on position

Fourier Transform Definition

- Assume f is a function on $\mathcal{R}$ such that

$$\int_{-\infty}^{\infty} |f(t)|\, dt = \int_{\mathcal{R}} |f(t)|\, dt \quad \text{is convergent.}$$

- define $\hat{f}(\omega)$, for every real ω as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

- the function $\hat{f}(\omega)$ is known as the *Fourier Transform* of $f(t)$.

- Often in engineering notation $F(\omega)$ is used instead of $\hat{f}(\omega)$.

Linearity of the Fourier Transform

- The mapping $\mathcal{F} : f(t) \mapsto \hat{f}(\omega)$ is linear

  That is, with $\mathcal{F}[f(t)] = \hat{f}(\omega)$ and $\mathcal{F}[g(t)] = \hat{g}(\omega)$,

  then we have $\mathcal{F}[\alpha f(t) + \beta g(t)] = \alpha \hat{f}(\omega) + \beta \hat{g}(\omega)$.

Invertibility of the Fourier Transform

- The mapping $\mathcal{F} : f(t) \mapsto \hat{f}(\omega)$ can be inverted.

  That is, with $\mathcal{F}[f(t)] = \hat{f}(\omega)$,

  there is an inverse mapping $\mathcal{F}^{-1}[\hat{f}(\omega)] = f(t)$.

- We have the *Inverse Fourier Transform*

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} \, d\omega$$

The Discrete Fourier Transform

- Often we deal with sampled data
- Numerical computations require discrete data
- We need a Fourier Transform in these cases
  - ◇ a direct and obvious extension of Fourier Transform

The Discrete Fourier Transform (cont.)

- Consider a discrete data set of N measurements (or samples) as the column vector with elements x_i

$$x = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix}$$

- We can define the Discrete Fourier Transform of x as a column vector X with elements X_k

$$X = \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{N-1} \end{bmatrix}$$

- Where each X_k is defined as,

$$X_k = \frac{1}{N} \sum_{n=0}^{N-1} x_n \, e^{\frac{-i2\pi kn}{N}}$$

Discrete Fourier Transform & its Inverse

- Discrete Fourier Transform

$$X_k = \frac{1}{N} \sum_{n=0}^{N-1} x_n \, e^{\frac{-i2\pi kn}{N}}$$

- Inverse Discrete Fourier Transform

$$x_n = \sum_{k=0}^{N-1} X_k \, e^{\frac{i2\pi kn}{N}}$$

Discrete Fourier Transform as Inner Product

- Consider inner product $\langle x, \phi_k \rangle$
- where ϕ_k is the k^{th} Fourier basis vector

$$\phi_k = \begin{bmatrix} e^{\frac{i2\pi k0}{N}} \\ e^{\frac{i2\pi k1}{N}} \\ \vdots \\ e^{\frac{i2\pi k(N-1)}{N}} \end{bmatrix}$$

- We have then

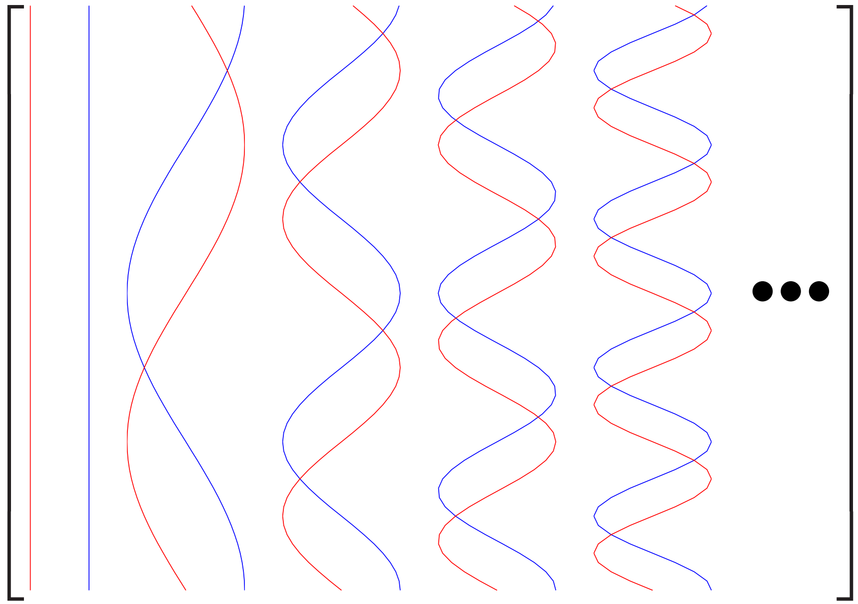
$$X_k = \frac{1}{N}\langle x, \phi_k \rangle = \frac{1}{N}\begin{bmatrix} x_0 & x_1 & \dots & x_{N-1} \end{bmatrix} \begin{bmatrix} e^{\frac{-i2\pi k0}{N}} \\ e^{\frac{-i2\pi k1}{N}} \\ \vdots \\ e^{\frac{-i2\pi k(N-1)}{N}} \end{bmatrix} = \frac{1}{N}\sum_{n=0}^{N-1} x_n \, e^{\frac{-i2\pi kn}{N}}$$

Discrete Fourier Transform as Matrix Transformation

$$[X_0 \quad X_1 \quad \dots \quad X_{N-1}]$$

$$= \frac{1}{N}[x_0 \quad x_1 \quad \dots \quad x_{N-1}] \begin{bmatrix} e^{\frac{-i2\pi\cdot 0\cdot 0}{N}} & e^{\frac{-i2\pi\cdot 1\cdot 0}{N}} & \dots & e^{\frac{-i2\pi\cdot(N-1)\cdot 0}{N}} \\ e^{\frac{-i2\pi\cdot 0\cdot 1}{N}} & e^{\frac{-i2\pi\cdot 1\cdot 1}{N}} & \dots & e^{\frac{-i2\pi\cdot(N-1)\cdot 1}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{\frac{-i2\pi\cdot 0\cdot(N-1)}{N}} & e^{\frac{-i2\pi\cdot 1\cdot(N-1)}{N}} & \dots & e^{\frac{-i2\pi\cdot(N-1)\cdot(N-1)}{N}} \end{bmatrix}$$

$$= \frac{1}{N}[x_0 \quad x_1 \quad \dots \quad x_{N-1}]$$



Discrete Fourier Transform as Coefficients and Basis

- Let $\{\phi_k\}_{k=0}^{N-1}$, be an orthonormal basis for V,
- then, u be an arbitrary vector in V, can be written as

$$u = \langle u, \phi_0 \rangle \phi_0 + \ldots + \langle u, \phi_{N-1} \rangle \phi_{N-1} = \sum_{k=0}^{N-1} \underbrace{\langle u, \phi_k \rangle}_{\text{coeffs}} \underbrace{\phi_k}_{\text{basis vector}}$$

- re-interpret Discrete Fourier Transform in this way,

$$x = \underbrace{\langle x, \phi_0 \rangle}_{X_0} \phi_0 + \underbrace{\langle x, \phi_1 \rangle}_{X_1} \phi_1 + \ldots + \underbrace{\langle x, \phi_{N-1} \rangle}_{X_{N-1}} \phi_{N-1} = \sum_{k=0}^{N-1} \underbrace{\langle x, \phi_k \rangle}_{\text{coeffs}} \underbrace{\phi_k}_{\text{basis vector}}$$

$$= x \text{ is a weighted sum of complex exponentials (sinusoids)}$$

An Intuitive Approach

Music as a signal

- What is music?
- What is sound?
- What is a signal?
- How do we think about signals?

Music as a point in space!

- Let’s consider a piece of music as a point in space:
  - ◇ a high dimensional space
  - ◇ how could we describe this point in space?
  - ◇ where is it?

Describing Things

How do we describe things in general?

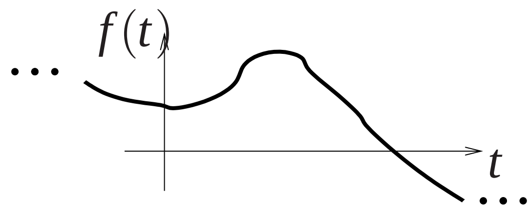
- Maybe we ...
  - ◇ describe the thing's various *components*
  - ◇ then describe how these components are *assembled* to make the thing.

... back to signals

- This is a useful concept for us in signals and systems
  - ◇ describe the signal by some *components* (**analysis**)
  - ◇ describe how the signal is *assembled* (**synthesis**)

Describing Music/Signals

- maybe in your mind you think of a signal as the *graph* of a function such as:



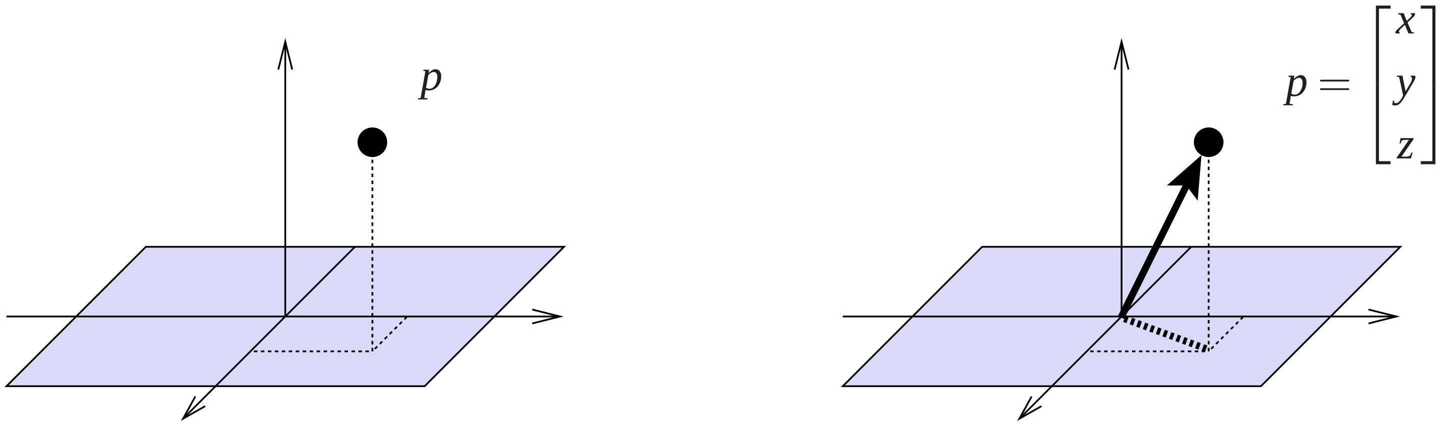
- ... but we think of a signal as a *point in a "high dimensional space"*!

Why do this?

- gives a concept of "closeness"
- gives a geometry ...
- geometry blends mathematics with *intuition*

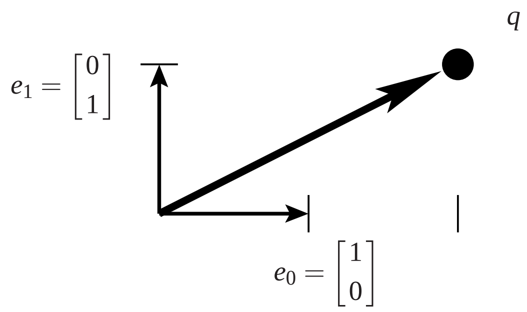
Point in Space = Vector

- Example, point in 3D



Favorite Point q in 2D

- consider



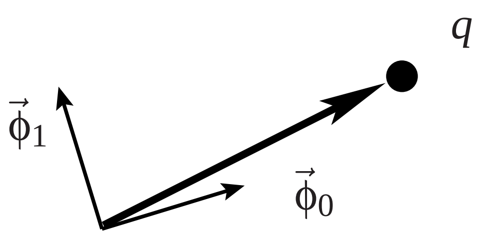
- Here the point q, can be represented as the vector

$$\begin{aligned} q &= a_0 \vec{e}_0 + a_1 \vec{e}_1 \\ &= 2\vec{e}_0 + 1\vec{e}_1 \\ &= 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{aligned}$$

- This description uses *coefficients* a_0 and a_1 with respect to *basis vectors* $\vec{e}_0$ and $\vec{e}_1$.

Another look at favorite point q in 2D

- consider again

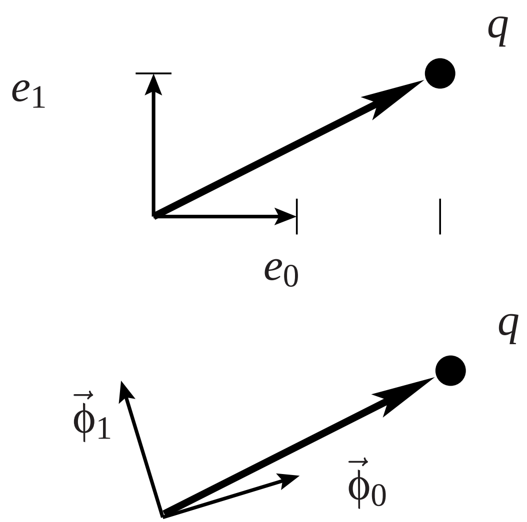


- Here the same point q, can be represented in a different way

$$q = \beta_0 \vec{\phi}_0 + \beta_1 \vec{\phi}_1$$

- This description uses *coefficients* β_0 and β_1 with respect to *basis vectors* $\vec{\phi}_0$ and $\vec{\phi}_1$.

We have *two* ways of describing our favorite point q



$$q = a_0\,\vec{e}_0 + a_1\,\vec{e}_1$$

$$q = \beta_0\,\vec{\phi}_0 + \beta_1\,\vec{\phi}_1$$

- The *coefficients* depend on which basis set you use.
- Which is the better basis set? ...
- ... it depends what you're trying to do!

The way you describe things, depends on what you choose as your components!

What does this have to do with Music?

- consider 10 seconds of CD-audio (samples of music)
- this is a discrete-time signal: $x[0]$, $x[1]$, . . ., $x[440999]$
- could think of as the vector (why?)

$$\mathbf{x} = \begin{bmatrix} x[0] \\ x[1] \\ \cdots \\ x[440,999] \end{bmatrix}$$

- this is a point in a 441,000 dimensional space

What does this have to do with Music? (cont.)

- We've seen the coefficients to describe a vector depend on the *basis vector* set we choose.
- If we choose the **time samples** as the basis vectors we could represent our music as

$$\mathbf{x} = x[0]\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix} + x[1]\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} + \ldots + x[440,999]\begin{bmatrix}0\\0\\0\\\vdots\\1\end{bmatrix}$$

$$= x[0]\,\vec{e}_0 + x[1]\,\vec{e}_1 + \ldots + x[440,999]\,\vec{e}_{440,999}$$

- we have chosen a *different* basis set, say $\{\phi_0, \phi_1, \ldots, \phi_{440,999}, \}$,

What does this have to do with Music? (cont.)

- Remember the coefficients to describe a vector depend on the *basis vector* set we choose.
- using the natural basis vectors (times samples) $\{e_0, e_1, \ldots, e_{440{,}999}, \}$, we have

$$\mathbf{x} \;=\; x[0]\,\vec{e}_0 \;+\; x[1]\,\vec{e}_1 \;+\; \ldots + x[440,999]\,\vec{e}_{440{,}999} = \sum_{i=0}^{440{,}999} x[i]\,\vec{e}_i$$

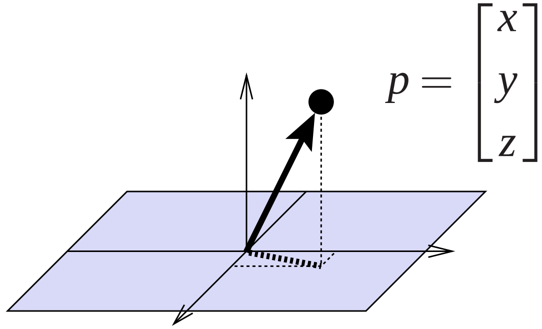
- using the basis vectors $\{\phi_0,\ \phi_1,\ \ldots,\ \phi_{440{,}999}, \}$, we have

$$\mathbf{x} \;=\; \beta_0\,\vec{\phi}_0 \;+\; \beta_1\,\vec{\phi}_1 \;+\; \ldots + \beta_{440{,}999}\,\vec{\phi}_{440{,}999} = \sum_{i=0}^{440{,}999} \beta_i\,\vec{\phi}_i$$

- note: the coefficients $x[i]$ and β_i would be different

Think of approximation

- Example, point in 3D, (take biggest coordinates)



- Example, music using natural basis (*Name that tune!*)
  - take biggest samples ... same as taking biggest coefficients with respect to basis set $\{e_0, e_1, \ldots, e_{440,999}\}$
- Example, music again, but using **Fourier basis**, (*Name that tune!*)
  - ...same as taking biggest coefficients with respect to basis set $\{\phi_0, \phi_1, \ldots, \phi_{440,999}\}$

We GET TO CHOOSE our basis set!

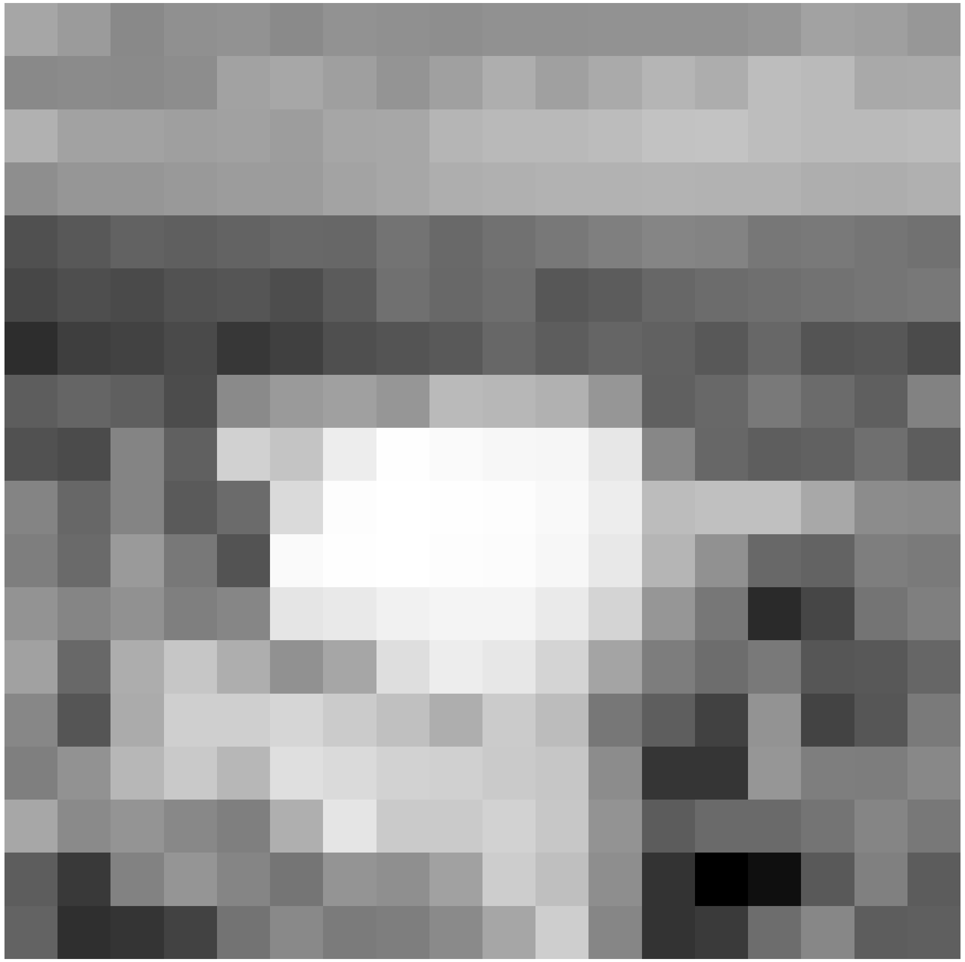
Name that tune!

- The idea of the game: I give you a partial description of song, you try to guess what song it is

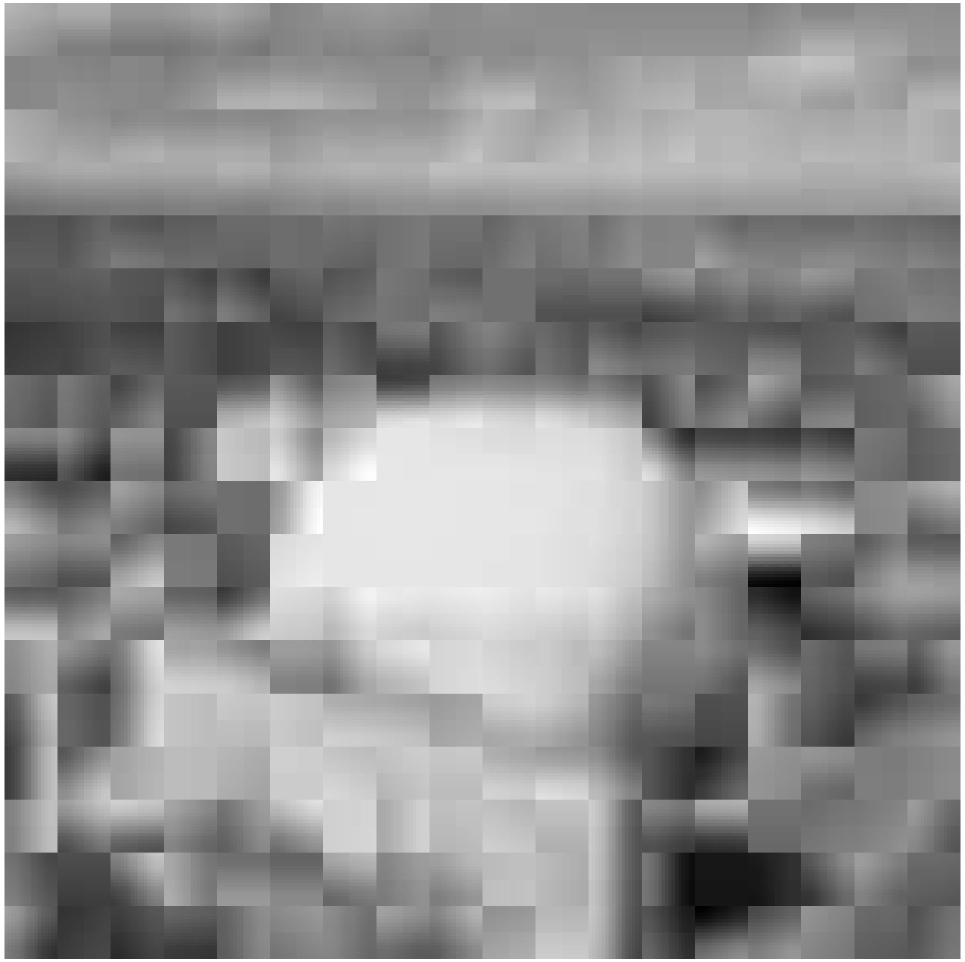
- One way to play: I give you a partial description in the <u>time domain</u>:
  - ◇ the biggest time-sample
  - ◇ the 10 biggest time-samples
  - ◇ the 100 biggest time-samples ...

- Another way: I give you a partial description in the <u>frequency domain</u>:
  - ◇ the biggest sine wave component
  - ◇ the 10 biggest sine wave components
  - ◇ the 100 biggest sine wave components ...

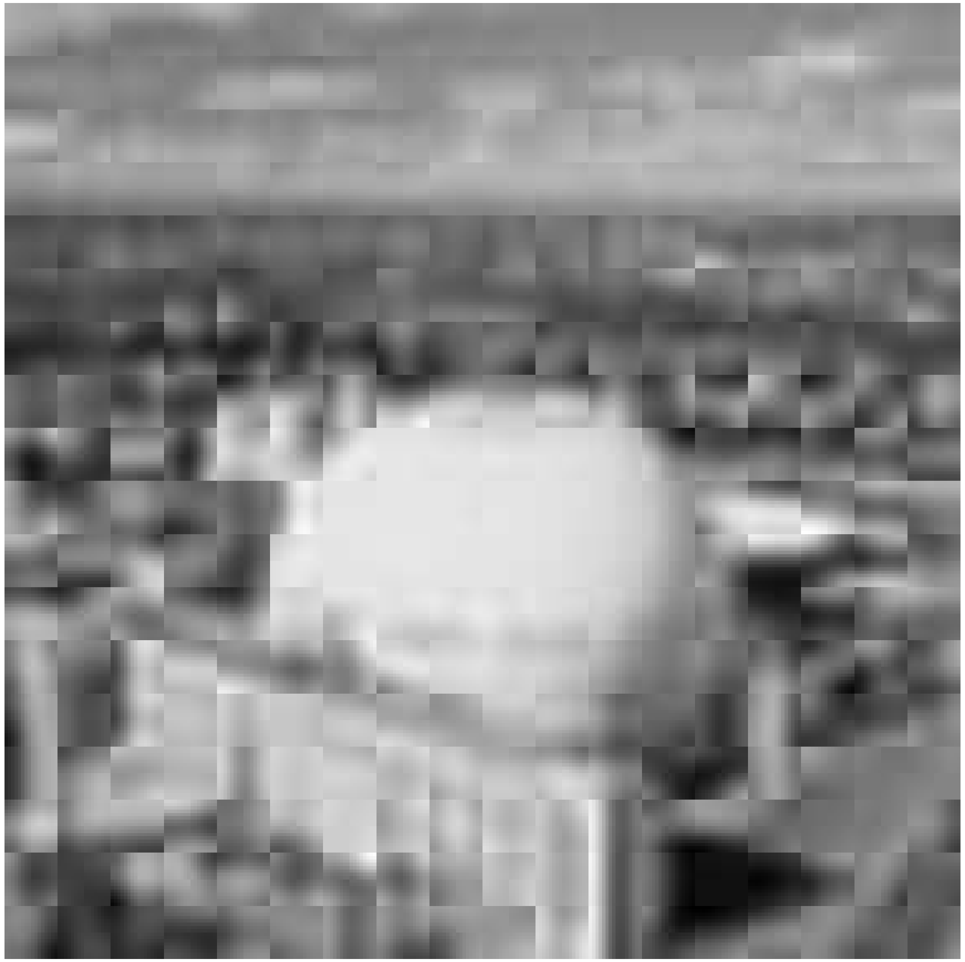
Not just used for sounds...but IMAGES too!



Not just used for sounds...but IMAGES too!



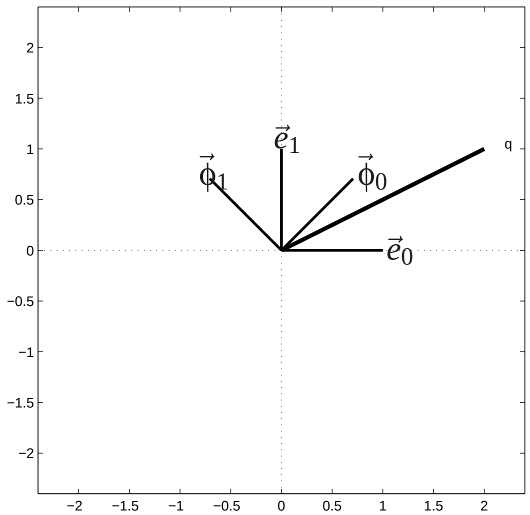
Not just used for sounds...but IMAGES too!



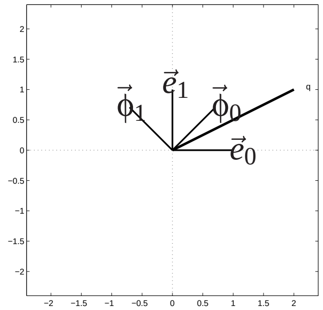
Not just used for sounds...but IMAGES too!

Points in Space: 2-D

- Given a vector in $2D$ and a orthonormal set of basis vectors $\vec{\phi}_0$ and $\vec{\phi}_1$, how do we find the coordinates?
- Given $\vec{q} = 2\vec{e}_0 + 1\vec{e}_1$, what are coordinates of $\vec{q}$ w. r. t. basis $\vec{\phi}_0, \vec{\phi}_1$?
- for the case $\vec{e}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{\phi}_0 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, and $\vec{\phi}_1 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ it looks like this



Finding the coefficients (*ii*)



- Use *dot product* (also known as *inner product*)
- Given $\vec{q} = 2\vec{e}_0 + 1\vec{e}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, we seek β_0 and β_1, such that $\vec{q} = \beta_0 \vec{\phi}_0 + \beta_1 \vec{\phi}_1$.

$$\beta_0 = \vec{q} \cdot \vec{\phi}_0 = \vec{q}^{\,T} \vec{\phi}_0 = [2 \quad 1] \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{3}{\sqrt{2}} \approx 2.12$$

$$\beta_1 = \vec{q} \cdot \vec{\phi}_1 = \vec{q}^{\,T} \vec{\phi}_1 = [2 \quad 1] \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{-1}{\sqrt{2}} \approx 0.707$$

Inner product (*i*)

- to find coefficient β_0, took inner product of $\vec{q}$ with basis vector $\vec{\phi}_0$.

$$\beta_0 = \vec{q} \cdot \vec{\phi}_0 = \vec{q}^{\,T} \vec{\phi}_0 = [2 \quad 1] \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \sum_{i=0}^{1} q(i)\phi_0(i)$$

- if our vector instead lived in in N dimensions? Concept is the same...

$$\beta_0 = \vec{q} \cdot \vec{\phi}_0 = \sum_{i=0}^{N-1} q(i)\phi_0(i)$$

- If our vector was *infinite dimensional* ...concept is the same

$$\beta_0 = \vec{q} \cdot \vec{\phi}_0 = \sum_{i=0}^{\infty} q(i)\phi_0(i)$$

- If our vector was *infinite dimensional* and defined on interval $0 < t < T$ ...concept is the same

$$\beta_0 = \vec{q} \cdot \vec{\phi}_0 = \int_0^T q(t)\phi_0(t)\, dt$$

Fourier Transform is same Concept!

- The Fourier Transform can be used to represent
- Fourier Transform uses the orthog basis "vectors" over $-\infty < t < \infty$
  - $e^{j\omega t} = \cos(\omega t) + i \sin(\omega t)$
- Fourier Transform "coefficients" found by inner products

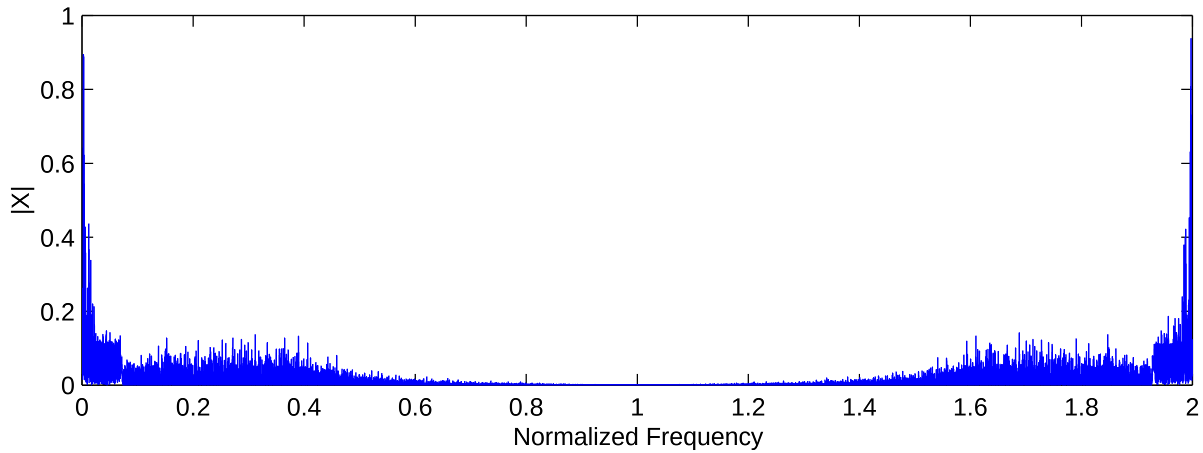
$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt \\ &= \mathcal{F}\{x(t)\} \end{aligned}$$

Fourier Transform

- Let's us think either in frequency domain or time domain
- Fourier Transform: $X(\omega) = \mathcal{F}\{x(t)\}$
- Inverse Fourier Transform: $x(t) = \mathcal{F}^{-1}\{X(\omega)\}$
- Fourier Transform Pair: $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$

An Audio Example

- Look at the Fourier Transform of 7 seconds audio shown below:



- Can you "see" how it should sound?
  - ◇ it's a bit of a problem
  - ◇ we see how much of each sinusoid (each 7 seconds long) is needed to make the audio...

... but that's not how we (as humans) interpret the sound!

Variation on the Theme

Frequency content over a shorter period of time.

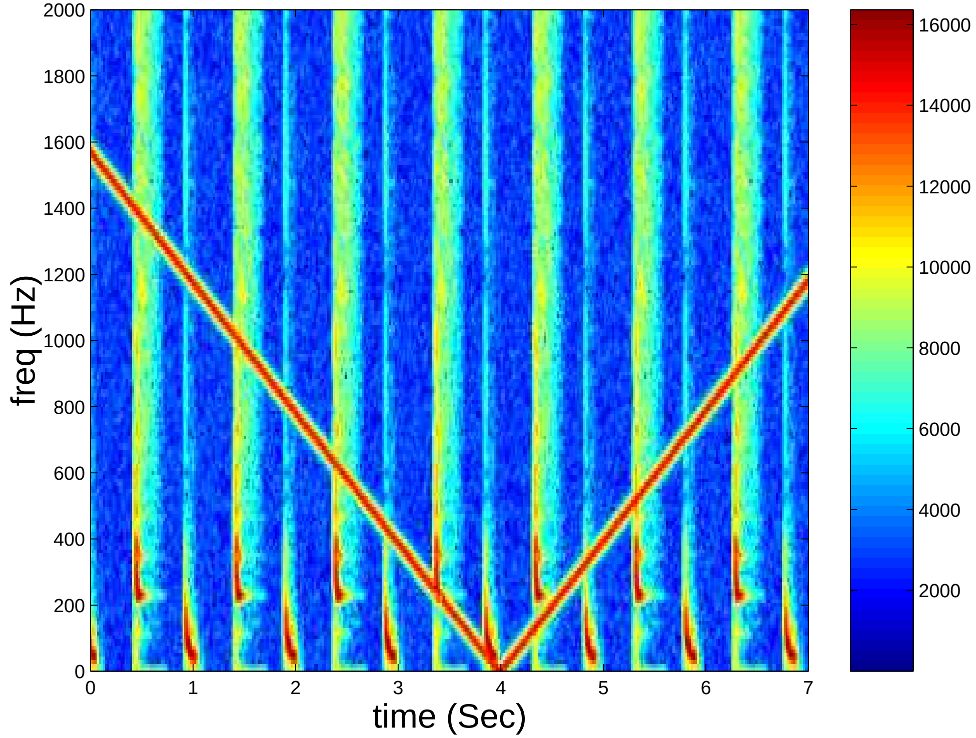
- Why not segment the audio into short time frames?
- Perform the FFT analysis on each short segment
- This is called the **periodogram**

Create an image by

- Placing each FFT vertically
- Encode large $|X|$ values as "hot"
- Can now see evolution of spectra, as time progesses

Audio Example Revisited

- A Periodogram of our audio signal



- What can we understand about signal now?
- Let's listen to it!

Periodogram

- By segmenting our data, we can "see" time evolution of the spectra.
- We obtained this new capability by:
  - ◇ moving away from using a set of "global basis function" (defined over the time duration of the signal)...
  - ◇ ... to a set of basis functions that are active only in "local regions"

This periodogram leads to ideas found in wavelets

- Wavelets are an analysis of signals (functions)
  - ◇ where the basis functions have compact support
  - ◇ have particular scaling properties

You'll be seeing wavelets with Niklas Grip in the next course section

Thanks for your attention!